Reminiscences

Ken-iti Sato (*Nagoya*)

I was born in June 1934 in Tokyo, and studied mathematics in University of Tokyo. My adviser was Kôsaku Yosida. The first mathematical paper was in 1961. The directions of my works since then were as follows:

(i) Boundary problems of Markov processes, 1962–1971, Tokyo.
(ii) Operator semigroups and Banach lattices, 1967–1972, Tokyo.
(iii) Infinitely divisible distributions, Lévy processes, additive processes, Ornstein-Uhlenbeck type processes, and writing of the CUP book, 1973–1999, Tokyo, Kanazawa, Nagoya.
(iv) Population-genetic models, 1975–1980. Tokyo, Kanazawa.
(v) Further on infinitely divisible distributions and Lévy processes, 2000–2019, Nagoya.

In 1994, I made a recollection talk[1] at the age of sixty. Completion of the CUP book mentioned in (iii) was later. Today I would like to talk mainly about the book, that is, *Lévy Processes and Infinitely Divisible Distributions* with xii+486 pages published by Cambridge University Press in 1999. It is the most widely known among my mathematical contributions. In fact, according to MathSciNet, the book has 1882 citations (including those of the revised edition of 2013) as of October 19. According to Google Scholar, it has 5733 citations. They are scattered in many areas of science. Lévy processes are very popular in these two decades in the areas using probability theory; J. Bertoin's book[2] also has lots of citations. As I mentioned only shortly in the preface why I wrote the book, I would talk here about my motivation behind the book.

The oldest and most important object of study in probability theory is partial sums of independent identically distributed random variables $S_n = Z_1 + \cdots + Z_n$,

This is a revised version of the article that appeared in Cooperative Research Report 434 (March 2020) Institute of Statistical Mathematics, Tachikawa, Tokyo, Japan, pp. 87–101. It is an extension of a talk given at the institute on November 8, 2019.

[1]The English translation is *Banach lattices, potential operators, population-genetic models, L distributions, and Lévy processes* in my home page http://ksato.jp/ (Selected Miscellaneous Writings).

[2]*Lévy Processes*, Cambridge University Press, 1996.

$n = 1, 2, \ldots$, which we call a random walk. The next most important object would be Brownian motion, or Wiener's measure on the space of continuous functions. I agree with D.W. Stroock saying in his book[3], "Wiener's measure is quite possibly the single most important object in all of modern probability theory". Then what is the third most important object in probability theory? In my opinion, it is the class of Lévy processes. A *Lévy process* is defined to be a stochastic process $\{X_t \colon t \geq 0\}$ on the d-dimensional Euclidean space $\mathbb{R}^d$ which is with stationary independent increments, starting at the point 0, continuous in probability, and satisfying the cadlag condition (i.e. almost surely each path $X_t(\omega)$ is right-continuous with left limits $X_{t-}(\omega)$ in t). If we do not require the cadlag condition, then we call $\{X_t \colon t \geq 0\}$ *Lévy process in law.* However, any Lévy process in law has a modification[4] which is a Lévy process. Hence, sometimes we use the word Lévy process in the meaning of Lévy process in law.

Let $\mu_1 * \mu_2$ denote the convolution of two distributions μ_1, μ_2 on $\mathbb{R}^d$. For a distribution μ and a positive integer n let μ^{n*} denote $\mu * \cdots * \mu$ (n factors). A distribution μ is called *infinitely divisible* if for each n there is a distribution μ_n such that $\mu = {\mu_n}^{n*}$. Let $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$, be the characteristic function $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx)$ with $\langle z, x\rangle$ being the inner product of z and x in $\mathbb{R}^d$. Then μ is infinitely divisible if and only if for each n there is a distribution μ_n such that $\widehat{\mu}(z) = \widehat{\mu}_n(z)^n$.

The class of Lévy processes $\{X_t \colon t \geq 0\}$ on $\mathbb{R}^d$ corresponds with the class of infinitely divisible distributions μ on $\mathbb{R}^d$ one-to-one[5] and onto. The correspondence is described by $\mu = \mathcal{L}(X_1)$, the distribution (or law) of X_1.

The characteristic function of an infinitely divisible distribution μ on $\mathbb{R}^d$ is described by a unique triplet (A, ν, γ) of a $d \times d$ symmetric nonnegative-definite matrix A, a measure ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(dx) < \infty$ with $|x| = \langle x, x\rangle^{1/2}$, and $\gamma \in \mathbb{R}^d$ in the form

$$\widehat{\mu}(z) = \exp\left[-\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\leq 1\}}(x)\right)\nu(dx)\right],$$

where $1_{\{|x|\leq 1\}}(x) = 1$ or 0 according as $|x| \leq 1$ or $|x| > 1$. Conversely, for any triplet (A, ν, γ), there exists a unique infinitely divisible distribution μ whose characteristic function is described by (A, ν, γ). This description of $\widehat{\mu}(z)$ is called *Lévy–Khintchine*

[3] *Probability Theory, an Analytic View*, Cambridge University Press, 1993; see page xi.

[4] $\{X'_t \colon t \geq 0\}$ is called a modification of $\{X_t \colon t \geq 0\}$ if $P[X'_t = X_t] = 1$ for every t. Here and in the following, P stands for probability and E for expectation.

[5] Here we identify two Lévy processes if they have a common system of finite-dimensional distributions.

representation of an infinitely divisible distribution μ. Here A, ν, and γ are respectively called *Gaussian covariance*, *Lévy measure*, and *location parameter* of μ.

If $\{X_t \colon t \geq 0\}$ is a Lévy process corresponding to μ, then $\mu_t = \mathcal{L}(X_t)$ is infinitely divisible and the triplet of μ_t is $(tA, t\nu, t\gamma)$. The word Lévy–Khintchine representation of μ is also used as of $\{X_t\}$.

The Lévy–Khintchine representation is proved in the knowledge of the second course analysis. The class of Lévy processes is thus an easily describable class. Brownian motion on $\mathbb{R}^d$ is a Lévy process with triplet $(I, 0, 0)$, where I is the $d \times d$ identity matrix. Poisson process is a Lévy process on $\mathbb{R}$ with triplet $(0, \delta_1, 1)$, where δ_1 is the distribution[6] concentrated at 1. In spite of the simplicity of the definition, the class of Lévy processes is surprisingly rich. A lot of diverse processes with complicated jumps are included in the class by the freedom of choice of Lévy measure. For example, the jumps of the path of a Lévy process $\{X_t\}$ on $\mathbb{R}$ with $\nu((-\infty, 0)) = 0$ and $\int_{(0,1]} x\nu(dx) = \infty$ are countable and satisfy $X_t - X_{t-} > 0$ but, for any $t_1 < t_2$, $\sum_{t_1 < t \leq t_2} (X_t - X_{t-}) = \infty$ and $P[\, X_t < 0\,] > 0$ for all $t > 0$ as if there is drift of size $-\infty$. The class of Lévy processes has inexhaustible features connected with jumps of random motion, by which I am tempted to call it the third most important object in probability theory. It is like Brownian motion has inexhaustible features connected with continuous random motion.

Researches up to 1990s accumulated many results on properties of Lévy processes and infinitely divisible distributions. So I had the following two points in the purposes of the book.

Point I. Give an introduction to the theory of Lévy processes with minimum prerequisite knowledge. Furnish all theorems with detailed proofs.

Point II. Based on the correspondence of Lévy processes and infinitely divisible distributions, develop theories of the two objects simultaneously.

For Point I, I did not assume the advanced theory of stochastic processes with continuous time parameter. I assumed the knowledge roughly equivalent to W. Feller's book[7], measure theory, and several results in the cited books in probability or analysis. For instance the correspondence of Lévy processes and infinitely divisible distributions is based on Kolmogorov's extension theorem. I aimed at writing a readable book (or, at least, a book such that a fair amount of its contents are readable), as there was no such book for Lévy processes. I was one of the last generation of students that

[6] A distribution concentrated at a point c is denoted by δ_c and called a trivial distribution.

[7] *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd ed., Wiley, 1968.

learned probability theory without martingales. A stochastic process is an additive process[8] if and only if it is a spatially homogeneous Markov process; it is a Lévy process if and only if it is a temporally and spatially homogeneous Markov process. This approach to Lévy and additive processes is a natural one and makes the proofs of a basic part of the theory fairly accessible. From this thought I avoided the use of martingale theory and used Markov processes with continuous time parameter.

I had to make a big exception for Point II. That is decomposition theory of infinitely divisible distributions developed by Cramér, Linnik, Ostrovskii, Cuppens, R. Shimizu, and others. As this theory seems to have no connection with Lévy processes, I did not include it in the book.

Further I had many more points in my purposes. Let me describe them in Points III to X.

Following Paul Lévy, we call a stochastic process $\{X_t\colon t \geq 0\}$ an *additive process* if it is with (not necessarily stationary) independent increments, starting at 0, continuous in probability, and satisfying the cadlag condition. If we do not require the cadlag condition, we call it an *additive process in law*. But, similarly to the case of Lévy process, we sometimes use the word *additive process* in the meaning of additive process in law. A result of Lévy and Khintchine shows that $\mu_{s,t} = \mathcal{L}(X_t - X_s)$ is infinitely divisible for $0 \leq s \leq t$ if $\{X_t\}$ is an additive process in law and that the class of additive processes in law corresponds one-to-one onto the class of families $\{\mu_{s,t}\colon 0 \leq s \leq t < \infty\}$ of infinitely divisible distributions satisfying[9] $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$, $\mu_{s,s} = \delta_0$, $\mu_{s,t} \to \delta_0$ as $s \uparrow t$, and $\mu_{s,t} \to \delta_0$ as $t \downarrow s$. Hence the theory of additive processes is entirely within the theory of infinitely divisible distributions. Let (A_t, ν_t, γ_t) be the triplet of $\mathcal{L}(X_t)$. If $\{X_t\}$ is a Lévy process, then $(A_1, \nu_1, \gamma_1) = (A, \nu, \gamma)$ and $(A_t, \nu_t, \gamma_t) = (tA, t\nu, t\gamma)$. I had the following point.

Point III. Develop the basic part of the theory of an additive process $\{X_t\}$. This includes the Lévy–Itô decomposition of paths of $\{X_t\}$. Here (independently scattered) Poisson random measure on $H = (0,\infty) \times (\mathbb{R}^d \setminus \{0\})$ with intensity measure $\widetilde{\nu}$ defined by $\widetilde{\nu}((0,t] \times B) = \nu_t(B)$ for any Borel set B is introduced; each jump $(s, X_s(\omega) - X_{s-}(\omega))$ is considered as a point in H; this clarifies the probabilistic meaning of ν_t.

[8] The definition will be given a bit later.

[9] Convergence of a sequence of distributions μ_n to a distribution μ, denoted by $\mu_n \to \mu$, means that $\int f(x)\mu_n(dx)$ tends to $\int f(x)\mu(dx)$ for all bounded continuous functions f.

In fact, in Chapter 4, I gave a new simpler proof of the Lévy–Itô decomposition, based on the idea that it is enough to construct the additive process satisfying the asserted decomposition, starting from the triplet (A_t, ν_t, γ_t) of $\{X_t\}$.

A distribution μ on $\mathbb{R}^d$ is called *stable* if, for any positive integer n, there are $a_n > 0$ and $c_n \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z)^n = \widehat{\mu}(a_n z)e^{i\langle c_n, z\rangle}$; μ is called *strictly stable* if it is stable with $c_n = 0$. If μ is stable, then it is infinitely divisible. A Lévy process $\{X_t\}$ with $\mathcal{L}(X_1)$ being stable or strictly stable is called a stable or strictly stable process, respectively. They were studied in the first stage of the development of the theory in 1920s. A distribution μ on $\mathbb{R}^d$ is called *selfdecomposable* if, for any $b > 1$, there is a distribution ρ_b satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho_b}(z)$. If μ is selfdecomposable, then μ and ρ_b are infinitely divisible and ρ_b is uniquely determined by b and μ. Selfdecomposable distributions were discovered by Lévy in 1936 in answer to a problem posed by Khintchine, who called them distributions of the class L; the name *selfdecomposable* is due to Loève. A Lévy process $\{X_t\}$ with $\mathcal{L}(X_1)$ being selfdecomposable is called a selfdecomposable process.

Point IV. Give an introduction to the theory of stable and selfdecomposable processes, including processes called semi-stable and semi-selfdecomposable.

Stable distributions and processes are treated in several books, but comprehensive treatment of selfdecomposable distributions and proceeses was not found earlier. A stochastic process $\{Y_t : t \geq 0\}$ is called *selfsimilar* if, for every $b > 0$, there is $a > 0$ such that[10] $\{Y_{bt} : t \geq 0\} \stackrel{\mathrm{d}}{=} \{aY_t : t \geq 0\}$. A stochastic process is a selfsimilar Lévy process if and only if it is a strictly stable process. Any non-trivial[11] stable process $\{X_t : t \geq 0\}$ has a unique *index* $\alpha \in (0, 2]$ such that, for any $b > 0$, $\{X_{bt} : t \geq 0\} \stackrel{\mathrm{d}}{=} \{b^{1/\alpha}X_t + c_b(t) : t \geq 0\}$ with a non-random function $c_b(t)$.

Three characterizations of selfdecomposability had been known; first as the limit distribution of summation (in some way) of a sequence of independent (not always identically distributed) random variables, second by its Lévy measure $\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)k_\xi(r)r^{-1}dr$, where S is the unit sphere $\{|\xi| = 1\}$ in $\mathbb{R}^d$, λ is a finite measure on S, and $k_\xi(r)$ is nonnegative and decreasing[12] in r and measurable in ξ,

[10]We write $\{X_t : t \geq 0\} \stackrel{\mathrm{d}}{=} \{Y_t : t \geq 0\}$ if the two processes have an identical system of finite-dimensional distributions.

[11]A Lévy process is called *trivial* if, for each t, $\mathcal{L}(X_t)$ is trivial.

[12]A function $f(t)$ is called increasing or decreasing, respectively, if $f(t_1) \leq f(t_2)$ for all $t_1 < t_2$ or if $f(t_1) \geq f(t_2)$ for all $t_1 < t_2$.

and third as the limit of an Ornstein–Uhlenbeck type process[13] convergent in distribution[14] as $t \to \infty$. I found, in [1990b S] and [1991 S], the fourth characterization as the distribution at a fixed time of a selfsimilar additive process. This characterization drew attention of people in mathematical finance; they named selfsimilar additive process as Sato process. In the development of my interest in selfsimilarity M. Maejima was influential.

Let me recall M. Yamazato's result[15] that all selfdecomposable distributions on $\mathbb{R}$ are unimodal. It was an ingenious solution of the long-standing problem. I was stimulated very much by this work of his, which was done in 1976. In 1965 I moved from Tokyo Metropolitan University to a newly created department of Tokyo University of Education (I think a better translation of the name is Tokyo Pedagogical University). It was essentially a second department of mathematics of that university. Four professors in probability gathered there: G. Maruyama and M. Motoo as full professors and M. Fukushima and I as associate professors. A. Shimizu was one of assistants. Yamazato came in as a graduate student and studied with me branching processes and infinitely divisible distributions. The progress made by Yamazato and later by T. Watanabe[16] encouraged me to work further. Selfdecomposable distributions were objects of research of Yamazato and me for several years: a detailed study of the shape of their density on $\mathbb{R}$ in [1978 S–Yz] and [1981 S–Yz], and then a study of their relation to Ornstein–Uhlenbeck type processes on $\mathbb{R}^d$ in [1984 S–Yz]. I tried for some years to show that all non-degenerate selfdecomposable distributions on $\mathbb{R}^d$, $d \geq 2$, are absolutely continuous[17] (their Lévy measures may be continuous singular); I succeeded in its proof in [1982 S] by a new method. What is an extension to $\mathbb{R}^d$, $d \geq 2$, of the unimodality of selfdecomposable distributions on $\mathbb{R}$? We do not know an adequate definition of unimodality for $d \geq 2$ yet.

Point V. Indicate time evolution of distributional properties of a Lévy process in qualitative nature. (Dependence of μ_t on t in Chapters 5 and 10)

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[13] A stochastic process $\{Y_t : t \geq 0\}$ is called an Ornstein–Uhlenbeck type process if $Y_t = J + Z_t - c\int_0^t Y_s ds$, where $\{Z_t\}$ is a Lévy process, $c > 0$, and J is a random variable independent of $\{Z_t\}$. Sometimes it is called an Ornstein–Uhlenbeck process driven by a Lévy process $\{Z_t\}$.

[14] The process $\{Y_t\}$ is convergent in distribution as $t \to \infty$ if and only if the Lévy measure ν_Z of $\{Z_t\}$ satisfies $\int_{|x|>1} \log |x| \, \nu_Z(dx) < \infty$.

[15] Unimodality of infinitely divisible distribution functions of class L. Ann. Probab., 6 (1978), 523–531.

[16] This is Toshiro Watanabe. There are other probabilists with name T. Watanabe.

[17] The words "absolutely continuous" and "singular" are with respect to Lebesgue measure unless otherwise noted.

The distribution $\mu_t = \mathcal{L}(X_t)$ of a Lévy process $\{X_t\}$ is linear with respect to t in some sense, because $\log \widehat{\mu}_t(z)$ is linear with respect to t. However, a general Lévy process can have time evolution in the qualitative properties of μ_t. This had not been well-known, probably because stable processes, being thought as typical examples of Lévy processes, do not have time evolution in the qualitative distributional properties. Nevertheless, a pioneering work of H. Rubin[18] gave an idea to show a remarkable result in this direction, that is, given an arbitrary increasing function $f(t)$ from $[0,\infty)$ to $[0,1]\cup\{\infty\}$, one can find a Lévy processes on $\mathbb{R}$ such that $f(t)$ equals the infimum of the Hausdorff dimensions of all Borel sets B with $\mu_t(B)=1$ if μ_t is singular and equals ∞ if μ_t is absolutely continuous. In [1994 S] I pointed out the existence of a Lévy process on $\mathbb{R}$ such that μ_t is continuous singular for $t<1$ and absolutely continuous for $t\geq 1$; by a slight change of construction, μ_1 becomes continuous singular. This was an extension of a paper[19] of H.G. Tucker, who followed Rubin's method. It looks like a phase transition in statistical physics.

The distribution $\mu_t(dx) = (c^t/\Gamma(t))x^{t-1}e^{-cx}1_{(0,\infty)}(x)dx$ on $\mathbb{R}$ with $t>0$ and $c>0$ is called Γ-distribution with shape parameter t and scale parameter c; μ_1 is called exponential distribution with parameter c. Then μ_t is infinitely divisible and the Lévy process $\{X_t\}$ corresponding to μ_1 satisfies $\mathcal{L}(X_t)=\mu_t$; hence $\{X_t\}$ is called Γ-process. This μ_t is unimodal with mode being 0 for $t\leq 1$ and $(t-1)/c$ for $t>1$. Here we see a mild time evolution (from mode 0 to positive mode). I was interested in the existence of a Lévy process on $\mathbb{R}$ in which time evolution in modality (unimodal to multimodal, or reverse, or repetition of such changes) is observed, and wrote a paper [1995 S] on this subject.

Concerning the time evolutions for $d=1$ of the two types above (one is from continuous singular to absolutely continuous and the other is in modality) T. Watanabe solved many problems in 1994–2000 and our knowledge greatly expanded.

Point VI. Subordination and density transformation of Lévy processes on $\mathbb{R}^d$. They are two important transformations from Lévy processes to Lévy processes.

Transformation called *subordination* is time change of a Lévy process $\{X_t\}$ on $\mathbb{R}^d$ to a Lévy process $\{Y_t\}$ on $\mathbb{R}^d$ in the form $Y_t = X_{Z_t}$ by an increasing Lévy process $\{Z_t\}$ independent of $\{X_t\}$. It was studied and named by S. Bochner in 1949. Thus

[18] Supports of convolutions of identical distributions, *Proc. Fifth Berkeley Symp. Math. Stat. Probab.* Vol. 2, Part 1 (1967), 415–422.

[19] On a necessary and sufficient condition that an infinitely divisible distribution be absolutely continuous. Trans. Amer. Math. Soc. **118** (1965), 316–330.

an increasing Lévy process is called *subordinator*. A Lévy process $\{Z_t\}$ with Lévy measure ν_Z is a subordinator if and only if $\nu_Z((-\infty,0))=0$, $\int_{(0,1]} x\,\nu_Z(dx)<\infty$, and $\widehat{\mu}_Z(z)=\exp\left[i\beta z+\int_0^\infty(e^{izx}-1)\nu_Z(dx)\right]$ with $\beta\geq 0$, where $\mu_Z=\mathcal{L}(Z_1)$. In a section of Chapter 6, I made a concise introduction to subordination; this part is frequently referred to.

Let $\mathbf{D}$ be the space of mappings $\omega(t)$ from $[0,\infty)$ into $\mathbb{R}^d$ right-continuous with left limits. Write $x_t(\omega)=\omega(t)$. Let $\mathcal{F}_t$ or $\mathcal{F}_{\mathbf{D}}$, respectively, be the smallest σ-algebra that makes $\{x_s\colon s\in[0,t]\}$ or $\{x_s\colon s\geq 0\}$ measurable. Then any Lévy process is realized by some probability measure P on $\mathcal{F}_{\mathbf{D}}$ as $\{x_t\colon t\geq 0\}$. The restriction of P to $\mathcal{F}_t$ is denoted by $[P]_{\mathcal{F}_t}$. The class of Lévy processes on $\mathbb{R}^d$ thus corresponds with a class of P on $\mathcal{F}_{\mathbf{D}}$. Let P and $P^\sharp$ be two probability measures in this class. Then $P^\sharp\perp P$ (i.e. $P^\sharp$ is singular with respect to P) if $P^\sharp\neq P$. If $[P^\sharp]_{\mathcal{F}_t}\ll[P]_{\mathcal{F}_t}$ (i.e. $[P^\sharp]_{\mathcal{F}_t}$ is absolutely continuous with respect to $[P]_{\mathcal{F}_t}$) holds for some $t>0$, then it holds for all $t>0$. If $[P^\sharp]_{\mathcal{F}_t}\perp[P]_{\mathcal{F}_t}$ holds for some $t>0$, then it holds for all $t>0$. Let $[P^\sharp]_{\mathcal{F}_t}\approx[P]_{\mathcal{F}_t}$ denote mutual absolute continuity of $[P^\sharp]_{\mathcal{F}_t}$ and $[P]_{\mathcal{F}_t}$. Then, using the Hellinger–Kakutani distance between the Lévy measures of P and $P^\sharp$, we can give a necessary and sufficient condition for $[P^\sharp]_{\mathcal{F}_t}\approx[P]_{\mathcal{F}_t}$ in terms of their triplets; the Radon–Nikodým derivative $d[P^\sharp]_{\mathcal{F}_t}/d[P]_{\mathcal{F}_t}$ is expressed as the exponential of a Lévy process on $\mathbb{R}$. Thus, starting from Lévy process P, we can construct all Lévy processes $P^\sharp$ such that $[P^\sharp]_{\mathcal{F}_t}\approx[P]_{\mathcal{F}_t}$. This is called *density transformation*. Esscher transformation is a simple example, where we have, with some $\eta\in\mathbb{R}^d$ and $c\in\mathbb{R}$, $P^\sharp(B)=e^{-ct}\int_B e^{\langle\eta,x_t(\omega)\rangle}P(d\omega)$ for $B\in\mathcal{F}_t$. Further, we can extend the theory to the Lebesgue decomposition of $[P^\sharp]_{\mathcal{F}_t}$ with respect to $[P]_{\mathcal{F}_t}$, starting from any P and $P^\sharp$. See Section 60 of the revised edition of the book and my lecture notes[20].

Point VII. Introduction to recurrence-transience criteria for Lévy processes and related behaviors as $t\to\infty$.

A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is called *recurrent* if $\liminf_{t\to\infty}|X_t|=0$ a.s. (= almost surely); it is called *transient* if $|X_t|\to\infty$ as $t\to\infty$, a.s. Any $\{X_t\}$ is either recurrent or transient. If $d\geq 3$, then any genuinely d-dimensional Lévy process is transient. Hence the criterion problem of recurrence-transience is for $d=1$, 2. Two criteria are known: (1) the potential measure of order 0, finite on any compact set, defined by $V(B)=\int_0^\infty\mu_t(B)dt$, does not exist or exists; (2) the function[21] $\operatorname{Re}(-1/\log\widehat{\mu}(z))$

[20] *Density Transformation in Lévy Processes* (2000) Lecture Notes, No. 7, MaPhySto, Centre for Math. Physics and Stochastics, Univ. Aarhus. You can download it from the site of MaPhySto.

[21] Here Re stands for the real part (of a complex number).

is non-integrable or integrable on a bounded neighborhood of 0. The latter is by K.L. Chung–W.H. Fuchs, F. Spitzer, and S.C. Port–C.J. Stone. However, no criterion is known in terms of the triplet (A, ν, γ). Only in symmetric 1-dimensional case, Shepp's criterion is known. A related result is that, in 1-dimensional transient case, there are three cases: $X_t \to \infty$ a.s., or $X_t \to -\infty$ a.s., or the set of limit points as $t \to \infty$ is $\{-\infty, \infty\}$ a.s. (oscillating case), for which a criterion is known in terms of Lévy measure (H. Kesten, K.B. Erickson). I intended to make comprehensive treatment with examples in Chapter 7, but some results were without proof.

Point VIII. Introduction to potential theory for Lévy processes on $\mathbb{R}^d$.

This is treated in Chapter 8. The essential idea is the strong Markov property. The basic concepts and results were developed in G.A. Hunt's papers in 1957 and 58 for a class of temporally homogeneous Markov processes later formulated in the book by R.M. Blumenthal and R.K. Getoor[22] as standard processes or (more strongly) Hunt processes. We had to discuss Conditions (ACT) and (ACP), absolute continuity of transition measures and potential measures, respectively. Lévy processes are Hunt processes, but the capacity theory of Hunt applies only under Condition (ACP). The capacity theory for general Lévy processes was established by the Port–Stone papers in 1971. In order to fully discuss the materials, I need sophisticated prerequisite. Instead I chose to give only main results with proofs. Thus, only for F_σ-sets B, the hitting time $T_B = \inf\{t > 0 \colon X_t \in B\}$ of B was shown to be a stopping time, q-capacitary measure of B and q-capacity $C^q(B)$ for $q > 0$ were defined, and its relation with q-energy $e^q(B)$ was analyzed. We have $C^q(B) = 0$ if and only if B is essentially polar. Applications to stable processes by S. Orey and M. Kanda were given. Although I did not use the fine topology induced by a Lévy process, the classification of Lévy processes related to properties of hitting times and regularity of a point by Kesten in 1969 was given. Writing of this chapter was not easy.

Point IX. Introduction to Wiener–Hopf factorizations for Lévy processes on $\mathbb{R}$.

Let $\{X_t\}$ be a Lévy process on $\mathbb{R}$ with $\mu_t = \mathcal{L}(X_t)$ and $\mu = \mu_1$ and let $V^q(B) = E\left[\int_0^\infty e^{-qt}\, 1_B(X_t)\, dt\right] = \int_0^\infty e^{-qt}\mu_t(B)dt$ for $q > 0$. Then qV^q is infinitely divisible and its characteristic function $q(q - \log\widehat{\mu}(z))^{-1}$ is written as $\varphi_q^+(z)\,\varphi_q^-(z)$, the product of the two characteristic functions $\varphi_q^+(z)$ and $\varphi_q^-(z)$ of some infinitely divisible distributions on $[0, \infty)$ and $(-\infty, 0]$, respectively. Decompositions having a resemblance to this are known for the joint distribution of the processes $M_t = \sup_{s \le t} X_s$,

[22] *Markov Processes and Potential Theory*, Academic Press, 1968.

$Y_t = M_t - X_t$, and $\Lambda_t = \inf\{s \in [0,t]\colon \max\{X_s, X_{s-}\} = M_t\}$ and also for that of the processes $R_t = T_{(t,\infty)}$ and $\Gamma_t = X_{R_t} - t$. These beautiful results are called Wiener–Hopf factorizations. They are used in the analysis of short-time or long-time behaviors of $\{X_t\}$. Another application is to the analysis of Lévy processes without positive jumps. Although an elegant treatment is known as in Bertoin's book, I took a road beginning with compound Poisson processes and approximating a general Lévy process by them. This road is close to the historical development and we can look at the connection with Wiener–Hopf technique in complex analysis.

Point X. Give as many explicit examples as possible.

Development up to 1990s supplied lots of new examples of infinitely divisible distributions: Student's t, Fisher's F, Gumbel, Weibull (with $0 < \alpha \leq 1$), Pareto, log-normal, logistic, half-Cauchy, generalized inverse Gaussian, hyperbolic, log of Γ-variable (E 18.19), hyperbolic-cosine (Lévy's stochastic area), distributions involving Bessel functions, and so on. Also new subclasses of the class of infinitely divisible distributions were introduced, such as GGC (genaralized Γ-convolutions), gcmed (generalized convolutions of mixtures of exponential distributions), L_m, L_∞ (subclasses of the class L of selfdecomposable distributions), and so on. In one dimension, see the book[23] by F.W. Steutel and K. van Harn. Those examples show the richness of the theory of Lévy processes and infinitely divisible distributions.

I worked in University of Minnesota two academic years 1967/68 and 1975/76 as a visiting member; there were many probabilists, S. Orey, W.E. Pruitt, B.E. Fristedt, N. Jain, R.H. Cameron, G. Kallianpur, R.V. Chacon, among them. I worked also in University of Illinois one academic year 1968/69; there were J.L. Doob, F. Knight, D.L. Burkholder, W. Philipp. From seminars in Minnesota and Illinois and from personal contact I learned very much. Usually my work was half research and half teaching, but sometimes full teaching. In the year 1975/76 I had a one year course on stochastic processes for graduate students every Mon., Wed., Fri. and, in winter and spring quarters, lectures on calculus for undergraduate students in addition, four times a week. Such a long course on stochastic processes for graduate students does not exist in Japan; I enjoyed its preparation.

My contact with Doob began 1968 in Illinois. Every Tuesday after a probability seminar people went together to a pizza house for lunch. Doob always seemed enjoying talking and discussing. In one of such occasions, he suggested me to write a book

[23] *Infinite Divisibility of Probability Distributions on the Real Line*, Marcel Dekker, 2004.

on Markov processes on boundary in the series "Ergebnisse der Mathmatik und ihrer Grenzgebiete" of Springer-Verlag. He was one of the editors of the series. Two months after I accepted his suggestion. Then I was asked by him whether I would make a contract with Springer to write the book. I chose not to make a contract. In fact Doob advised me to make a contract, saying that when he intended to write a book *Stochastic Processes* for John Wiley, he signed the contract because his wife Elsie advised him to have the obligation. Then I began preparation for a comprehensive book. But the book was not realized, as the work was much harder than I thought and my interest shifted in several years. However, the planning of such a book had a good effect to me in the 1970s; I began thorough study of the books including Blumenthal–Getoor, Feller's Volume 2, and Spitzer[24] to have a solid basic knowledge.

I moved from Tokyo Univ. of Educ. to Kanazawa Univ. in 1976. People say that Tokyo Univ. of Educ. moved 70 kilometers and had a new name Tsukuba Univ. But this is incorrect. At first the relocation of the university was proposed about 1967 but there was strong opposition by professors and students. The university got into turmoil and, in 1969, even the entrance examination was canceled. The struggle ended with the closedown of Tokyo Univ. of Educ. and the creation of a new university called Tsukuba with new professors added. Many faculty members who opposed the relocation chose not to join Tsukuba Univ. They include the four professors in probability that I mentioned. After seven years at Kanazawa Univ., I moved to College of General Education, Nagoya Univ. in 1983. About ten years later, organizational changes were made in almost all national universities in Japan and I belonged to Faculty of Informatics. I resigned from Nagoya Univ. in 1996, two years before the retirement age, and started working freely at home, using library facility at the university. Actually I am a professor emeritus, but in Japan it is almost nominal.

The book from Cambridge was published under the contract on translation right between Kinokuniya Publishers and Cambridge Univ. Press. I wrote the original book in Japanese published in 1990 with title *Kahou Katei*, whose direct translation is *additive process*. I used the word in the meaning of Lévy process of today. So, I translated my book and simultaneously made a complete revision; the title was changed to Lévy Processes and Infinitely Divisible Distributions and, roughly speaking, the contents were doubled; in fact, all treatment of additive processes was new, semi-stability and semi-selfdecomposability were new, the notion of time evolution of

[24] *Principles of Random Walk*, Van Nostrand, 1964.

qualitative distributional properties of a Lévy process was new, density transformation was new, Shepp's criterion of recurrence-transience in symmetric case was new, the chapter on potential theory for Lévy processes was entirely new, all exercises were newly inserted, and so on. This work started in 1993 and done mostly after my resignation from Nagoya University. Points I to X were of the book from Cambridge; they became gradually clear while I was making the complete revision.

During and after writing the book, I visited Zurich (Switzerland), Aarhus (Denmark), Guanajuato (Mexico), and other places for lectures and meetings. I made joint works with O.E. Barndorff-Nielsen, A. Lindner, M. Maejima, J. Pedersen, V. Pérez-Abreu, Y. Ueda, T. Watanabe, and K. Yamamuro. A. Rocha-Arteaga and I wrote a slim book *Topics in Infinitely Divisible Distributions and Lévy Processes*, which was published by Sociedad Matemática Mexicana in 2003. But, from January 2005 on, I could not go abroad for reasons of health. At home I made revision of the book from Cambridge. Revised edition with xiv+521 pages was published in 2013 in paperback; the original edition was preserved except some corrections and 30 pages of Supplement and new references were added. Recently Rocha-Arteaga and I made a thorough revision of our book and Revised edition was published by Springer in 2019.

Let me add two things here. One is about early years while I was working in Tokyo and another is about some symposiums from 1969.

In the latter half of 1960s Maruyama and the probability group in the second mathematics department of Tokyo Univ. of Educ. acted as the center of Japanese mathematicians' movement against USA in the Vietnam War. It was connected with an international movement of mathematicians with other centers being L. Schwartz in France, S. Smale in USA, and E.B. Dynkin in USSR. This, together with the conflict on the relocation problem of the university, took our energy. Nevertheless, works such as the excursion theory by Motoo and the theory of processes with infinitely divisible joint distributions by Maruyama were done there.

Now let me turn to recollection of joint meetings with USSR. Symposiums in Probability Theory (and Mathematical Statistics) between Japan and USSR were held seven times from 1969 to 1995[25]. The beginning was a letter to Dynkin from Maruyama. USSR probabilists (including A.N. Kolmogorov, Yu.V. Prokhorov, A.N. Shiryaev, Ya.G. Sinai, and Dynkin) reacted to the letter by preparing the Habarovsk

[25] I wrote an article: Twenty-seven years in Japan-USSR and Japan-Russia Symposiums (in Japanese). *Sugaku*, Vol. 48, No. 4 (Nov. 1996), 425–431.

symposium, and continuation of joint meetings by alternate organization in the two countries was hoped for. In fact, the seventh in 1995 was called Japan–Russia as USSR was non-existent, but in essence it was between Japan and the former USSR. Among the seven, four were in USSR and three were in Japan. In chronological order, they were held at Habarovsk, Kyoto, Tashkent, Tbilisi, Kyoto, Kiev, Tokyo. The four proceedings from the second symposium to the fifth were published in Lecture Notes in Mathematics (Springer) and the two proceedings of the sixth and seventh were published by World Scientific, Singapore. I did not join the first one, but attended all others. The symposiums had three kinds of difficulties. First, for the people of USSR traveling abroad was possible only by special permission, which was often hard to get. Some persons such as Dynkin could not get exit visa. Second, in Japan at that time we could hardly have financial support in going to meetings abroad or in preparing international meetings inside the country. Third, some people in probability seminars in Japan were strongly against the USSR regime and they argued that such symposiums meant taking part in the discrimination and suppression in USSR, so that many times we had to have long discussions. Those difficulties were great but, nevertheless, many people took part in the symposiums; the third one at Tashkent had 51 participants from Japan and 136 from USSR; the fourth one at Tbilisi had 45 from Japan and 278 from USSR; the fifth one at Kyoto had 25 from USSR and 190 from Japan; the last one at Tokyo had 34 from the former USSR and 142 from Japan. Additional pleasure for us was the visit to Moscow and Leningrad. Kolmogorov made a talk at the fourth one (Tbilisi, 1982), but he passed away in 1987. After the sixth one (Kiev, 1991) we were invited to his former *dacha* in the suburb of Moscow; on the wall he had painted pictures of animals and plants. I was also invited to some conferences at Vilnius, where B. Grigelionis was working. I still have vivid memories of the USSR probabilists and their conditions of living and working.

Selected papers authored by K. Sato solely or jointly
(In the order of the year of publication)[26]

The names of authors are abbreviated as follows: BN=O.E.Barndorff-Nielsen, L= A.Lindner, M=M.Maejima, PA=V.Pérez-Abreu, Pd=J.Pedersen, Pn=Lei Pan, S=K.Sato, Ud=Y.Ueda, Un=T.Ueno, W=Toshiro Watanabe, Ym=K.Yamamuro, Yz=M.Yamazato.

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[26] See http://ksato.jp/ for the complete list.

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(August 25, 2020)