LEBESGUE DECOMPOSITION BETWEEN TWO PATH SPACE MEASURES INDUCED BY LÉVY PROCESSES

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This is a report of some results in the lecture notes [9].

1. Hellinger–Kakutani inner product and distance

As Kakutani [3] (1948), Brody [1] (1971), and Newman [5], [6] (1972, 73) have shown, the Hellinger–Kakutani inner product and distance are powerful tool in the problems of absolute continuity and singularity.

Given a measure μ and a nonnegative measurable function f, we denote by $f\mu$ the measure defined as

$$(f\mu)(B) = \int_B f d\mu.$$

Let ρ_1, ρ_2 be σ-finite measures on a measurable space $(\Theta, \mathcal{B})$. The following notation is used: $\rho_2 \ll \rho_1$ means that ρ_2 is absolutely continuous with respect to ρ_1; $\rho_2 \perp \rho_1$ means that ρ_2 and ρ_1 are mutually singular; $\rho_2 \approx \rho_1$ means that $\rho_2 \ll \rho_1$ and $\rho_1 \ll \rho_2$.

Definition 1.1. Let $0 < \alpha < 1$. The Hellinger–Kakutani inner product of ρ_1 and ρ_2 of order α is the measure $H_\alpha(\rho_1, \rho_2)$ defined by

$$H_\alpha(\rho_1, \rho_2) = \left(\frac{d\rho_1}{d\rho}\right)^\alpha \left(\frac{d\rho_2}{d\rho}\right)^{1-\alpha} \rho, \quad 0 < \alpha < 1, \tag{1.1}$$

where $\rho_1 \ll \rho$ and $\rho_2 \ll \rho$. It is independent of the choice of ρ. Sometimes we write

$$dH_\alpha(\rho_1, \rho_2) = (d\rho_1)^\alpha (d\rho_2)^{1-\alpha}. \tag{1.2}$$

The total mass of $H_\alpha(\rho_1, \rho_2)$ is written as

$$h_\alpha(\rho_1, \rho_2) = \int_\Theta dH_\alpha(\rho_1, \rho_2). \tag{1.3}$$

Remark 1.2. We have

$$H_\alpha(\rho_1, \rho_2) \le \alpha\rho_1 + (1-\alpha)\rho_2. \tag{1.4}$$

We have $\rho_1 \perp \rho_2$ if and only if $h_\alpha(\rho_1, \rho_2) = 0$.

Definition 1.3. Write

$$C(\rho_1) = C_\rho(\rho_1) = \left\{\theta \in \Theta \colon \frac{d\rho_1}{d\rho} > 0\right\}, \tag{1.5}$$

$$C(\rho_2) = C_\rho(\rho_2) = \left\{\theta \in \Theta \colon \frac{d\rho_2}{d\rho} > 0\right\}, \tag{1.6}$$

where $\rho_1 \ll \rho$ and $\rho_2 \ll \rho$. $C(\rho_1)$ is the carrier of ρ_1 relative to ρ and $C(\rho_2)$ is the carrier of ρ_2 relative to ρ

Remark 1.4. Let $\rho_2 = {\rho_2}^{ac} + {\rho_2}^{s}$ be the Lebesgue decomposition of ρ_2 with respect to ρ_1, where ${\rho_2}^{ac}$ is absolutely continuous and ${\rho_2}^{s}$ is singular with respect to ρ_1. Then,

$${\rho_2}^{ac} = 1_{C(\rho_1)}\rho_2 \quad \text{and} \quad {\rho_2}^{s} = 1_{C(\rho_1)^c}\rho_2. \tag{1.7}$$

If ρ_1 and ρ_2 are finite, then

$$\lim_{\alpha\downarrow 0} h_\alpha(\rho_1, \rho_2) = \rho_2(C(\rho_1)) = {\rho_2}^{ac}(\Theta), \tag{1.8}$$

$$\lim_{\alpha\uparrow 1} h_\alpha(\rho_1, \rho_2) = \rho_1(C(\rho_2)). \tag{1.9}$$

Definition 1.5. Let $0 < \alpha < 1$. Define

$$K_\alpha(\rho_1, \rho_2) = \alpha\rho_1 + (1-\alpha)\rho_2 - H_\alpha(\rho_1, \rho_2), \tag{1.10}$$

which is a σ-finite measure. The total mass

$$k_\alpha(\rho_1, \rho_2) = \int_\Theta dK_\alpha(\rho_1, \rho_2) \tag{1.11}$$

is called the Hellinger–Kakutani distance of order α between ρ_1 and ρ_2.

Remark 1.6. Sometimes we write

$$dK_{1/2}(\rho_1, \rho_2) = \tfrac{1}{2}(\sqrt{d\rho_1} - \sqrt{d\rho_2})^2. \tag{1.12}$$

Let $\|\rho_1 - \rho_2\|$ be the total variation norm of $\rho_1 - \rho_2$, admitting infinity. Then

$$\|\rho_1 - \rho_2\| \geq 2\, k_{1/2}(\rho_1, \rho_2). \tag{1.13}$$

If ρ_1 and ρ_2 are finite measures, then

$$\|\rho_1 - \rho_2\| \leq c\, k_{1/2}(\rho_1, \rho_2)^{1/2}, \tag{1.14}$$

where $c = 2(\rho_1(\Theta) + \rho_2(\Theta))^{1/2}$.

Lemma 1.7. *Assume that*

$$k_\alpha(\rho_1, \rho_2) < \infty \tag{1.15}$$

for some $0 < \alpha < 1$. *Then* (1.15) *is true for all* $0 < \alpha < 1$ *and*

$$\lim_{\alpha \downarrow 0} k_\alpha(\rho_1, \rho_2) = \rho_2(C(\rho_1)^c) < \infty, \tag{1.16}$$

$$\lim_{\alpha \uparrow 1} k_\alpha(\rho_1, \rho_2) = \rho_1(C(\rho_2)^c) < \infty. \tag{1.17}$$

Lemma 1.8. *For* $j = 1, 2$, *let* ν_j *be* σ*-finite measures on* $\mathbb{R}^d$ *satisfying* $\nu_j(\{0\}) = 0$ *and* $\int_{\mathbb{R}^d}(|x|^2 \wedge 1) d\nu_j < \infty$. *If* $k_\alpha(\nu_1, \nu_2) < \infty$, *then*

$$\int_{|x| \le 1} |x| d|\nu_1 - \nu_2| < \infty, \tag{1.18}$$

$$\int_{|x| \le 1} |x| d|\nu_j - H_\alpha(\nu_1, \nu_2)| < \infty, \qquad j = 1, 2. \tag{1.19}$$

2. General theory

Let $\mathbf{D} = D([0, \infty), \mathbb{R}^d)$, $X_t(\omega) = \omega(t)$ for $\omega \in \mathbf{D}$, $\mathcal{F}_t^0 = \sigma(X_s\colon 0 \le s \le t)$, and $\mathcal{F}^0 = \sigma(X_s\colon 0 \le s < \infty)$. Any Lévy process on $\mathbb{R}^d$ can be realized as $(\{X_t\}, P)$, where P is a probability measure on $(\mathbf{D}, \mathcal{F}^0)$. It is said to have the generating triplet (A, ν, γ) if

$$E^P[e^{i\langle z, X_t\rangle}] = \exp\Big[t\Big(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}(x))\nu(dx)\Big)\Big] \tag{2.1}$$

for $z \in \mathbb{R}^d$, where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. (Lévy–Khintchine representation)

Let $(\{X_t\}, P)$ be a Lévy process with generating triplet (A, ν, γ), where P is a probability measure on $(\mathbf{D}, \mathcal{F}^0)$. For any $G \in \mathcal{B}_{(0,\infty)\times(\mathbb{R}^d \setminus \{0\})}$ let $J(G, \omega)$ be the number of $s > 0$ such that $(s, X_s(\omega) - X_{s-}(\omega)) \in G$. Then $J(G)$ has Poisson distribution with mean $\widetilde{\nu}(G)$, where $\widetilde{\nu} = ds \times \nu(dx)$. If $G_1, \dots, G_n$ are disjoint, then $J(G_1), \dots, J(G_n)$ are independent. We can define

$$X_t'(\omega) = \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times\{\varepsilon < |x| \le 1\}} \{xJ(d(s,x), \omega) - x\widetilde{\nu}(d(s,x))\} + \int_{(0,t]\times\{|x|>1\}} xJ(d(s,x), \omega), \tag{2.2}$$

where the convergence in the right-hand side is uniform in t in any finite time interval, P-a. s. Define

$$X_t''(\omega) = X_t(\omega) - X_t'(\omega). \tag{2.3}$$

Then, $(\{X'_t\}, P)$ and $(\{X''_t\}, P)$ are independent Lévy processes with generating triplets $(0, \nu, 0)$ and $(A, 0, \gamma)$, respectively. We call $(\{X'_t\}, P)$ and $(\{X''_t\}, P)$ the jump part and the continuous part of $(\{X_t\}, P)$, respectively. (Lévy–Itô decomposition)

Consider two Lévy processes $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ on $\mathbb{R}^d$, where P_1 and P_2 are probability measures on $(\mathbf{D}, \mathcal{F}^0)$. For $j = 1, 2$ denote the generating triplet of $(\{X_t\}, P_j)$ by (A_j, ν_j, γ_j). When $A_1 = A_2$, we write $A_1 = A_2 = A$. In this case define $\mathfrak{R}(A) = \{Ax \colon x \in \mathbb{R}^d\}$. Denote the restriction of P_j to $\mathcal{F}^0_t$ by P^t_j.

The following Theorem A was given by Newman [5], [6] (1972, 73). He essentially proved also Corollaries 2.1–2.5.

Theorem A. (i) *Suppose that*

$$\text{(NS)} \qquad k_\alpha(\nu_1, \nu_2) < \infty, \quad A_1 = A_2, \quad \textit{and} \quad \gamma_{21} \in \mathfrak{R}(A),$$

where

$$\gamma_{21} = \gamma_2 - \gamma_1 - \int_{|x| \le 1} x d(\nu_2 - \nu_1). \tag{2.4}$$

Then

$$H_\alpha(P^t_1, P^t_2) = e^{-tL_\alpha} P^t_\alpha \quad \textit{for } 0 < t < \infty,\ 0 < \alpha < 1, \tag{2.5}$$

where

$$L_\alpha = \tfrac{1}{2}\alpha(1-\alpha)\langle \eta, A\eta \rangle + k_\alpha(\nu_1, \nu_2) \tag{2.6}$$

with η satisfying $A\eta = \gamma_{21}$, and P_α is the probability measure for which $(\{X_t\}, P_\alpha)$ is the Lévy process generated by $(A, H_\alpha(\nu_1, \nu_2), \gamma_\alpha)$ with

$$\gamma_\alpha = \alpha\gamma_1 + (1-\alpha)\gamma_2 - \int_{|x| \le 1} x dK_\alpha(\nu_1, \nu_2). \tag{2.7}$$

(ii) *Suppose that* (NS) *is not satisfied, then*

$$H_\alpha(P^t_1, P^t_2) = 0 \quad \textit{for } 0 < t < \infty,\ 0 < \alpha < 1. \tag{2.8}$$

Corollary 2.1. *The following three conditions are equivalent.*

(1) *P^t_2 and P^t_1 are not mutually singular for some $0 < t < \infty$.*

(2) *P^t_2 and P^t_1 are not mutually singular for any $0 < t < \infty$.*

(3) *Condition* (NS) *is satisfied.*

Corollary 2.2. *If P^t_2 and P^t_1 are not mutually singular, then*

$$\nu_2(C(\nu_1)^c) < \infty \quad \textit{and} \quad \nu_1(C(\nu_2)^c) < \infty \tag{2.9}$$

and

$$P_2^t(C(P_1^t)) = e^{-t\nu_2(C(\nu_1)^c)} \quad \text{and} \quad P_1^t(C(P_2^t)) = e^{-t\nu_1(C(\nu_2)^c)}. \tag{2.10}$$

Corollary 2.3. *The following three conditions are equivalent.*

(1) $P_2^t \ll P_1^t$ *for some* $0 < t < \infty$.

(2) $P_2^t \ll P_1^t$ *for any* $0 < t < \infty$.

(3) $\nu_2 \ll \nu_1$ *and Condition* (NS) *is satisfied.*

Corollary 2.4. *The following three conditions are equivalent.*

(1) $P_2^t \approx P_1^t$ *for some* $0 < t < \infty$.

(2) $P_2^t \approx P_1^t$ *for any* $0 < t < \infty$.

(3) $\nu_2 \approx \nu_1$ *and Condition* (NS) *is satisfied.*

Corollary 2.5 (dichotomy)**.** *If* $\nu_2 \approx \nu_1$, *then either* $P_2^t \approx P_1^t$ *for all* $t > 0$ *or* $P_2^t \perp P_1^t$ *for all* $t > 0$.

The next corollary considers P_1 and P_2 on the whole $\mathcal{F}^0$.

Corollary 2.6. $P_2 \perp P_1$ *if* $P_2 \neq P_1$.

Let us prepare Theorem B. In the case where $P_2^t \approx P_1^t$, (i) and (iii) of Theorem B are trivial and (ii) was shown by Skorokhod [10], [11], [12] (1957, 60, 61) and Kunita and S. Watanabe [4] (1967), a proof of which is given in Sato [8] (1999). But here we do not assume $P_2^t \approx P_1^t$, nor $P_2^t \ll P_1^t$.

Let $P_2^t = (P_2^t)^{ac} + (P_2^t)^s$ be the Lebesgue decomposition of P_2^t with respect to P_1^t, and $\nu_2 = {\nu_2}^{ac} + {\nu_2}^s$ be the Lebesgue decomposition of ν_2 with respect to ν_1. Let $\nu = \nu_1 + \nu_2$. Choose the versions

$$\frac{d\nu_j}{d\nu} = f_j \qquad \text{for } j = 1,\, 2 \tag{2.11}$$

satisfying

$$f_1 \geq 0, \quad f_2 \geq 0, \quad \text{and } f_1 + f_2 = 1 \quad \text{everywhere on } \mathbb{R}^d. \tag{2.12}$$

Denote

$$\begin{cases} C_1 = \{f_1 = 1 \text{ and } f_2 = 0\}, \\ C_2 = \{f_1 = 0 \text{ and } f_2 = 1\}, \\ C = \{f_1 > 0 \text{ and } f_2 > 0\}. \end{cases} \tag{2.13}$$

Thus

$${\nu_2}^{ac} = 1_C \nu_2 \quad \text{and} \quad {\nu_2}^s = 1_{C_2}\nu_2 = 1_{C_1 \cup C_2}\nu_2 \tag{2.14}$$

and $d\nu_2{}^{ac}/d\nu_1$ has the following version:

$$\frac{d\nu_2{}^{ac}}{d\nu_1} = \begin{cases} f_2/f_1 & \text{on } C \\ 0 & \text{on } C_1 \cup C_2. \end{cases} \tag{2.15}$$

Define

$$g(x) = \begin{cases} \log(f_2/f_1) & \text{on } C \\ -\infty & \text{on } C_1 \cup C_2, \end{cases} \tag{2.16}$$

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{on } C \\ 0 & \text{on } C_1 \cup C_2. \end{cases} \tag{2.17}$$

Lemma 2.7. *Suppose that P_2^t and P_1^t are not mutually singular for $0 < t < \infty$. Then the following are true.*

(i) *We can define*

$$V_t = \lim_{\varepsilon \downarrow 0} \left(\sum_{(s, X_s - X_{s-}) \in (0,t] \times \{|x| > \varepsilon\}} \widetilde{g}(X_s - X_{s-}) - t \int_{|x|>\varepsilon} (e^{g(x)} - 1) \nu_1(dx) \right); \tag{2.18}$$

the right-hand side exists P_1-a. s. and the convergence is uniform on any bounded time interval P_1-a. s.

(ii) *Let $\eta \in \mathbb{R}^d$ and define*

$$U_t^{(\eta)} = \langle \eta, X_t'' \rangle - \frac{t}{2} \langle \eta, A\eta \rangle - t \langle \gamma_1, \eta \rangle + V_t, \tag{2.19}$$

where $\{X_t''\}$ is the continuous part of $(\{X_t\}, P_1)$. Then $\{U_t^{(\eta)} \colon t \geq 0\}$ is, under P_1, a Lévy process on $\mathbb{R}$ with generating triplet $(A_U^{(\eta)}, \nu_U, \gamma_U^{(\eta)})$ given by

$$A_U^{(\eta)} = \langle \eta, A\eta \rangle, \tag{2.20}$$

$$\nu_U(B) = \int_{\mathbb{R}^d} 1_B(g(x)) \nu_1(dx) \quad \text{for } B \in \mathcal{B}_{\mathbb{R} \setminus \{0\}}, \tag{2.21}$$

$$\gamma_U^{(\eta)} = -\tfrac{1}{2} \langle \eta, A\eta \rangle - \int_{\mathbb{R}^d} (e^{g(x)} - 1 - g(x) 1_{\{|g(x)| \leq 1\}}(x)) \nu_1(dx). \tag{2.22}$$

The processes $\{U_t^{(\eta)} \colon t \geq 0\}$ and $\{J((0,t] \times (C_1 \cup C_2)) \colon t \geq 0\}$ are independent under P_1.

Define $\Lambda_t \in \mathcal{F}_t^0$ by

$$\begin{aligned} \Lambda_t &= \{J((0,t] \times (C_1 \cup C_2)) = 0\} \\ &= \{X_s - X_{s-} \notin C_1 \cup C_2 \text{ for all } s \in (0,t]\}. \end{aligned} \tag{2.23}$$

To the best of our knowledge, the following results are new.

Theorem B. *Suppose that* P_2^t *and* P_1^t *are not mutually singular for* $0 < t < \infty$. *Then the following are true.*

(i) *For* $0 < t < \infty$ *the Lebesgue decomposition of* P_2^t *with respect to* P_1^t *is given by*

$$(P_2^t)^{ac} = 1_{\Lambda_t} P_2^t, \tag{2.24}$$

$$(P_2^t)^s = 1_{\mathbf{D}\setminus\Lambda_t} P_2^t. \tag{2.25}$$

We have $P_1(\Lambda_t) = e^{-t\nu_1(C_1)}$ *and* $P_2(\Lambda_t) = e^{-t\nu_2(C_2)}$.

(ii) *The Radon–Nikodým density of* $(P_2^t)^{ac}$ *is given by*

$$\frac{d(P_2^t)^{ac}}{dP_1^t} = e^{-t\nu_2(C_2)+U_t} 1_{\Lambda_t}, \tag{2.26}$$

where $U_t = U_t^{(\eta)}$ *with* η *satisfying* $A\eta = \gamma_{21}$.

(iii) *Let* Q *be the probability measure on* $(\mathbf{D}, \mathcal{F}^0)$ *for which* $(\{X_t\}, Q)$ *is the Lévy process with generating triplet* $(A, {\nu_2}^{ac}, \gamma_2 - \int_{|x|\le 1} x d{\nu_2}^s)$. *Then*

$$(P_2^t)^{ac} = e^{-t\nu_2(C_2)} Q^t. \tag{2.27}$$

Proofs of all results are given in [9].

3. Examples

1. *Gaussian case.* This is a special case of the results of Cameron and Martin. Suppose that $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ are Lévy processes on $\mathbb{R}^d$ with generating triplets $(A_1, 0, \gamma_1)$ and $(A_2, 0, \gamma_2)$, respectively. Then, for any fixed t, Theorems A and B give the following.

(i) The dichotomy holds: either $P_2^t \approx P_1^t$ or $P_2^t \perp P_1^t$.

(ii) $P_2^t \approx P_1^t$ if and only if

$$A_1 = A_2 \quad \text{and} \quad \gamma_2 - \gamma_1 \in \mathfrak{R}(A). \tag{3.1}$$

(iii) If $P_2^t \approx P_1^t$, then, for $0 < \alpha < 1$,

$$H_\alpha(P_1^t, P_2^t) = e^{-tL_\alpha} P_\alpha^t, \tag{3.2}$$

where P_α is the probability measure for which $(\{X_t\}, P_\alpha)$ is the Lévy process generated by $(A, 0, \gamma_\alpha)$ with $\gamma_\alpha = \alpha\gamma_1 + (1-\alpha)\gamma_2$, and

$$L_\alpha = \tfrac{1}{2}\alpha(1-\alpha)\langle \eta, A\eta \rangle \tag{3.3}$$

with η satisfying $A\eta = \gamma_2 - \gamma_1$.

(iv) If $P_2^t \approx P_1^t$, then

$$\frac{dP_2^t}{dP_1^t} = e^{U_t}, \tag{3.4}$$

where

$$U_t = \langle \eta, X_t \rangle - \tfrac{1}{2} t \langle \eta, A\eta \rangle - t\langle \gamma_1, \eta \rangle \tag{3.5}$$

with η satisfying $A\eta = \gamma_2 - \gamma_1$.

2. *Scaled Poisson processes with drift.* Suppose that both $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ are scaled Poisson processes with drift. That is, for $j = 1, 2$,

$$E^{P_j}[e^{izX_t}] = \exp\left[t\left(b_j(e^{ia_j z} - 1) + i\gamma_{0j} z\right)\right], \quad z \in \mathbb{R}, \tag{3.6}$$

with $b_j > 0$, $a_j \in \mathbb{R} \setminus \{0\}$, and $\gamma_{0j} \in \mathbb{R}$. Thus $\nu_j = b_j \delta_{a_j}$. This is the case studied by Dvoretzky, Kiefer, and Wolfowitz [2] (1953). P_2^t and P_1^t are not mutually singular if and only if $\gamma_{02} = \gamma_{01}$. Under the condition that $\gamma_{02} = \gamma_{01}$, there are two cases.

Case 1: $a_2 = a_1$. In this case we have $P_2^t \approx P_1^t$ and

$$P_2^t = (b_2/b_1)^{N_t} e^{-t(b_2 - b_1)} P_1^t, \tag{3.7}$$

where $N_t = N_t(\omega)$ is the number of jumps of $X_s(\omega)$ for $s \le t$.

Case 2: $a_2 \ne a_1$. In this case we have

$$(P_2^t)^{ac} = e^{t(b_1 - b_2)} 1_{\Lambda_t} P_1^t, \tag{3.8}$$

where $\Lambda_t = \{X_s - X_{s-} \ne a_1, a_2 \text{ for } s \in (0, t]\}$. Further we have $(P_2^t)^{ac}(\mathbf{D}) = e^{-tb_2}$ and $(P_2^t)^{ac} = e^{-tb_2} Q^t$, where $(\{X_t\}, Q)$ is a deterministic motion,

$$Q(X_t = t\gamma_{02} \text{ for } t \ge 0) = 1.$$

3. *Necessary conditions for* (NS). Suppose that P_2^t and P_1^t are not mutually singular for $0 < t < \infty$. Then, one can prove from Theorem A that the following three cases are possible and that no other cases can arise:

Case 1: $\nu_1(\mathbb{R}^d) < \infty$ and $\nu_2(\mathbb{R}^d) < \infty$.

Case 2: $\nu_1(\mathbb{R}^d) = \infty$, $\int_{|x|\le 1} |x| \nu_1(dx) < \infty$, and $\nu_2(\mathbb{R}^d) = \infty$, $\int_{|x|\le 1} |x| \nu_2(dx) < \infty$.

Case 3: $\int_{|x|\le 1} |x| \nu_1(dx) = \infty$ and $\int_{|x|\le 1} |x| \nu_2(dx) = \infty$.

Needless to say, these are not sufficient conditions for P_2^t and P_1^t not to be mutually singular.

4. *Lévy processes with finite Lévy measures.* Suppose that (A_j, ν_j, γ_j), $j = 1, 2$, satisfy $\nu_1(\mathbb{R}^d) < \infty$, $\nu_2(\mathbb{R}^d) < \infty$, $A_2 = A_1$, and $\gamma_{21} \in \mathfrak{R}(A)$. Note that $\gamma_{21} = \gamma_{02} - \gamma_{02}$,

where γ_{0j}, $j = 1, 2$, are respective drifts. We have

$$(P_2^t)^{ac}(\mathbf{D}) = e^{-t\nu_2^s(\mathbb{R}^d)}. \tag{3.9}$$

Thus, $P_2^t \ll P_1^t$ if and only if $\nu_2 \ll \nu_1$.

5. *Absolutely continuous change of Lévy measures.* We start from one Lévy process $(\{X_t\}, P_1)$ on $\mathbb{R}^d$ with generating triplet (A_1, ν_1, γ_1). Suppose that we are given a measurable function $g(x)$ with values $-\infty \le g(x) < \infty$ and a vector $\eta \in \mathbb{R}^d$. Assume that

$$\int_{\mathbb{R}^d} (e^{g(x)/2} - 1)^2 \nu_1(dx) < \infty. \tag{3.10}$$

Define (A_2, ν_2, γ_2) by

$$A_2 = A_1, \quad \nu_2(dx) = e^{g(x)}\nu_1(dx), \quad \gamma_2 = \gamma_1 + \int_{|x|\le 1} xd(\nu_2 - \nu_1) + A_1\eta.$$

Notice that (3.10) means that $k_{1/2}(\nu_1, \nu_2) < \infty$. Hence γ_2 is definable by Lemma 1.8. The condition (3.10) is equivalent to the property that

$$\int_{|g|\le 1} g^2 d\nu_1 + \int_{g>1} e^g d\nu_1 + \int_{g<-1} d\nu_1 < \infty. \tag{3.11}$$

It follows that

$$\int (1 \wedge |x|^2)\nu_2(dx) < \infty. \tag{3.12}$$

A new Lévy process $(\{X_t\}, P_2)$ with generating triplet (A_2, ν_2, γ_2) is obtained in this way. We have $P_2^t \ll P_1^t$ by Corollary 2.3 and $P_2^t = e^{U_t} 1_{\Lambda_t} P_1^t$ in the notation of Theorem B. This procedure to get $(\{X_t\}, P_2)$ from $(\{X_t\}, P_1)$ is called density transformation in [8] and [9]. Esscher transformation (or exponential transformation) in [7] and [8] is a special case. Drift transformation, deletion of jumps, and truncation of the support of Lévy measure in [9] are also special cases.

References

[1] Brody, E. J. (1971) An elementary proof of the Gaussian dichotomy theorem, *Zeit. Wahrsch. Verw. Gebiete* **20**, 217–226.

[2] Dvoretzky, A., Kiefer, J., and Wolfowitz, J. (1953) Sequential decision problems for processes with continuous time parameter. Testing hypotheses, *Ann. Math. Statist.* **24**, 254–264.

[3] Kakutani, S. (1948) On equivalence of infinite product measures, *Ann. Math.* **49**, 214–224.

[4] Kunita, H. and Watanabe, S. (1967) On square integrable martingales, *Nagoya Math. J.* **30**, 209–245.

[5] Newman, C. M. (1972) The inner product of path space measures corresponding to random processes with independent increments, *Bull. Amer. Math. Soc.* **78**, 268–271.

[6] Newman, C. M. (1973) On the orthogonality of independent increment processes, *Topics in Probability Theory* (ed. D. W. Stroock and S. R. S. Varadhan, Courant Inst. Math. Sci., New York Univ., New York), 93–111.
[7] Sato, K. (1990) Subordination depending on a parameter, *Probability Theory and Mathematical Statistics, Proc. Fifth Vilnius Conf.* (ed. B. Grigelionis et al., VSP/Mokslas, Utrecht/Vilnius) Vol. 2, 372–382.
[8] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press., Cambridge.
[9] Sato, K. (2000) *Density Transformation in Lévy Processes*, Lecture Notes, Centre for Math. Physics and Stochastics, Univ. Aarhus.
[10] Skorokhod, A. V. (1957) On the differentiability of measures which correspond to stochastic processes, I. Processes with independent increments, *Theory Probab. Appl.* **2**, 407–432.
[11] Skorokhod, A. V. (1960) On the differentiability of measures corresponding to random processes, II. Markov processes, *Theory Probab. Appl.* **5**, 40–49.
[12] Skorokhod, A. V. (1961) *Studies in the Theory of Random Processes,* Addison-Wesley, Reading, Mass., 1965. [Russian original published in 1961]

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