Memo November 29, 2007, from KS

A nested classes of the ranges of stochastic integrals with respect to Lévy processes on $\mathbb{R}^d$ can have the limit which is a proper subclass of $L_\infty(\mathbb{R}^d)$. This is shown by an example.

Theorem A. *If $\mu \in L_\infty(\mathbb{R}^d)$ with triplet (A,ν,γ), then ν has representation*

$$\nu(B) = \int_{(0,2)} \Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{1}$$

where

$$\begin{cases} \Gamma \text{ is a measure on } (0,2) \text{ satisfying } \int_{(0,2)} (\beta^{-1} + (2-\beta)^{-1}) \Gamma(d\beta) < \infty \text{ and} \\ \lambda_\beta \text{ is a probability measure on } S \text{ for each } \beta \text{ and measurable in } \beta. \end{cases} \tag{2}$$

This Γ is uniquely determined by ν, and λ_β is determined by ν up to β of Γ-measure 0. Conversely, given A, γ, Γ, and λ_β satisfying (2), we can find $\mu \in L_\infty(\mathbb{R}^d)$ with triplet (A,ν,γ), where ν satisfies (1).

This is Theorem 3.4 of [S80] or Theorem 22 of [RS03]. A similar fact was first discovered by Urbanik [U72, U73]. We call Γ the Γ-measure of $\mu \in L_\infty(\mathbb{R}^d)$, sometimes denoted by Γ^μ. If μ is Gaussian, then Γ^μ is zero measure.

Definition. Let $0 < \alpha < 2$. Let $L_\infty^{(\alpha)}(\mathbb{R}^d)$ denote the class of $\mu \in L_\infty(\mathbb{R}^d)$ with Γ^μ satisfying $\Gamma^\mu((0,\alpha]) = 0$.

The class $L_\infty^{(\alpha)}(\mathbb{R}^d)$ is closed under convolution, but is not closed under convergence.

Theorem B. *Let $0 < \alpha < 1$, $p(u) = u^{-\alpha-1}e^{-u}$, and $g(t) = \int_t^\infty p(u)du$ for $0 < t \leqslant \infty$. Let $t = f(s)$, $0 \leqslant s < \infty$, be defined by $s = g(t)$, $0 < t \leqslant \infty$. Define*

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^{\infty-} f(s) dX_s^{(\mu)} \right). \tag{3}$$

Then the domain and the range of Φ_f *are as follows:*

$$\begin{aligned}\mathfrak{D}(\Phi_f) &= \{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{|x|>1}|x|^\alpha\nu(dx)<\infty\}\\ &= \{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{\mathbb{R}^d}|x|^\alpha\mu(dx)<\infty\},\\ \mathfrak{R}(\Phi_f) &= \{\mu=\mu_{(A,\nu,\gamma)}\colon \nu(B)=\int_S\lambda(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha-1}h_\xi(u)du,\\ &\qquad B\in\mathcal{B}(\mathbb{R}^d), \textit{where } \lambda \textit{ is a measure on } S \textit{ and } h_\xi(u) \textit{ is a function}\\ &\qquad \textit{measurable in } \xi \textit{ and, for } \lambda\textit{-a. e. } \xi, \textit{ not identically zero,}\\ &\qquad \textit{completely monotone in } u\in(0,\infty), \textit{ and } \lim_{u\to\infty}h_\xi(u)=0.\}\end{aligned}$$

Moreover,

$$\mathfrak{D}(\Phi_f)=\{\mu\in I(\mathbb{R}^d)\colon \int_0^\infty |C_\mu(f(s)z)|ds<\infty \textit{ for } z\in\mathbb{R}^d\}.$$

This result is in Theorems 2.4 and 4.2 of [S06]. Note that, if $\alpha\geqslant 1$, then descriptions of the domain and the range are different from Theorem B.

Theorem C. *Let* $f(s)$ *and* Φ_f *be as in Theorem B. Let*

$$\mathfrak{R}_f^m=\mathfrak{R}_f^m(\mathbb{R}^d)=\Phi_f^m(\mathfrak{D}(\Phi_f^m)),\qquad m=1,2,\dots.$$

Then

$$I(\mathbb{R}^d)\supset\mathfrak{R}_f^1\supset\mathfrak{R}_f^2\supset\cdots, \tag{4}$$

$$\bigcap_{m=1}^\infty \mathfrak{R}_f^m=L_\infty^{(\alpha)}(\mathbb{R}^d). \tag{5}$$

Proof. Step 1. Let us show (4). Let $m\geqslant 1$. We have

$$\mathfrak{D}(\Phi_f^{m+1})=\{\mu\in\mathfrak{D}(\Phi_f^m)\colon \Phi_f^m(\mu)\in\mathfrak{D}(\Phi_f)\}=\{\mu\in\mathfrak{D}(\Phi_f)\colon \Phi_f(\mu)\in\mathfrak{D}(\Phi_f^m)\}.$$

Hence

$$\Phi_f(\mathfrak{D}(\Phi_f^{m+1}))\subset\mathfrak{D}(\Phi_f^m).$$

It follows that

$$\Phi_f^{m+1}(\mathfrak{D}(\Phi_f^{m+1}))\subset\Phi_f^m(\mathfrak{D}(\Phi_f^m)),$$

that is $\mathfrak{R}_f^{m+1}\subset\mathfrak{R}_f^m$.

Step 2. Let us show that

$$\mathfrak{R}_f^1\subset\mathcal{U}(I(\mathbb{R}^d))=U_0(\mathbb{R}^d). \tag{6}$$

Let $\mu \in \mathfrak{D}(\Phi_f)$ and $\widetilde{\mu} = \Phi_f(\mu)$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\nu}$, respectively. Let $(\lambda(d\xi), \nu_\xi(dr))$ be a polar decomposition of ν. Then

$$\widetilde{\nu}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) = \int_0^\infty p(t)dt \int_{\mathbb{R}^d} 1_B(tx)\nu(dx)$$

for $B \in \mathcal{B}(\mathbb{R}^d)$. If $B = \{r\xi : \xi \in D,\ r \in (s,\infty)\}$ with $D \in \mathcal{B}(S)$ and $s > 0$, then

$$\begin{aligned}
\widetilde{\nu}(B) &= \int_0^\infty p(t)dt \int_D \lambda(d\xi) \int_{s/t}^\infty \nu_\xi(dr) = \int_D \lambda(d\xi) \int_0^\infty \nu_\xi(dr) \int_{s/r}^\infty p(t)dt \\
&= \int_D \lambda(d\xi) \int_0^\infty r^{-1}\nu_\xi(dr) \int_s^\infty p(u/r)du \\
&= \int_D \lambda(d\xi) \int_s^\infty du \int_0^\infty p(u/r)r^{-1}\nu_\xi(dr).
\end{aligned}$$

Hence, letting $\widetilde{\lambda} = \lambda$ and

$$\widetilde{l}_\xi(u) = \int_0^\infty p(u/r)r^{-1}\nu_\xi(dr),$$

we obtain a polar decomposition $(\widetilde{\lambda}(d\xi), \widetilde{l}_\xi(u)du)$ of $\widetilde{\nu}$. Since p is decreasing, $\widetilde{l}_\xi(u)$ is decreasing in u. Therefore $\widetilde{\mu} \in U_0(\mathbb{R}^d)$.

Step 3. Let m be a positive integer. Let

$$v_m(t) = \frac{1}{(m-1)!} \int_t^1 \left(\log \frac{1}{t'}\right)^{m-1} dt', \qquad 0 \leqslant t \leqslant 1.$$

Then $v_m(0) = 1$. Let $t = u_m(s)$, $0 \leqslant s \leqslant 1$, be defined by $s = v_m(t)$, $0 \leqslant t \leqslant 1$. Let

$$\Phi_{u_m}(\mu) = \mathcal{L}\left(\int_0^1 u_m(s)dX_s^{(\mu)}\right).$$

Then $\mathfrak{D}(\Phi_{u_m}) = I(\mathbb{R}^d)$ and

$$\Phi_{u_m}(\mu) = \mathcal{U}^m(\mu), \qquad \mu \in I(\mathbb{R}^d). \tag{7}$$

Indeed, $\mathfrak{D}(\Phi_{u_m}) = I(\mathbb{R}^d)$, because the function $\widetilde{u}_m(s)$ defined by $\widetilde{u}_m(s) = u_m(s)$ for $0 \leqslant u \leqslant 1$ and $\widetilde{u}_m(s) = 0$ for $s > 1$ is locally square integrable. Since $v_1(t) = 1 - t$ and $u_1(s) = 1 - s$, (7) is true for $m = 1$. Suppose that (7) is true for a given m. Then

$$\begin{aligned}
C_{\mathcal{U}^{m+1}(\mu)}(z) &= \int_0^1 C_{\mathcal{U}^m(\mu)}(sz)ds = \int_0^1 ds \int_0^1 C_\mu(su_m(r)z)dr \\
&= \frac{1}{(m-1)!} \int_0^1 ds \int_0^1 C_\mu(stz) \left(\log \frac{1}{t}\right)^{m-1} dt \\
&= \frac{1}{(m-1)!} \int_0^1 ds \int_0^s C_\mu(tz) \left(\log \frac{s}{t}\right)^{m-1} \frac{dt}{s} \\
&= \frac{1}{(m-1)!} \int_0^1 C_\mu(tz)dt \int_t^1 \left(\log \frac{s}{t}\right)^{m-1} \frac{ds}{s}
\end{aligned}$$

$$= \frac{1}{m!}\int_0^1 C_\mu(tz)dt \left(\log\frac{1}{t}\right)^m dt,$$

which shows that $\Phi_{u_{m+1}}(\mu) = \mathcal{U}^{m+1}(\mu)$. Hence (7) is true for all m. This is essentially Jurek's result in [J04].

Step 4. Let m be a positive integer. Suppose that $\mu \in \mathfrak{D}(\Phi_f)$. Then $\mathcal{U}^m(\mu) \in \mathfrak{D}(\Phi_f)$ and $\Phi_f \mathcal{U}^m(\mu) = \mathcal{U}^m \Phi_f(\mu)$. Indeed, we have $\int_0^\infty |C_\mu(f(s)z)|ds < \infty$ by Theorem B. Hence

$$\begin{aligned}
&\int_0^\infty ds \int_0^1 |C_\mu(u_m(t)f(s)z)|dt \\
&\quad = \int_0^\infty p(s')ds' \int_0^1 |C_\mu(t's'z)|\frac{1}{(m-1)!}\left(\log\frac{1}{t'}\right)^{m-1} dt' \\
&\quad = \frac{1}{(m-1)!}\int_0^1 \left(\log\frac{1}{t'}\right)^{m-1} dt' \int_0^\infty |C_\mu(s'z)| p\left(\frac{s'}{t'}\right)\frac{ds'}{t'} \\
&\quad = \frac{1}{(m-1)!}\int_0^\infty |C_\mu(s'z)|(s')^{-\alpha-1}ds' \int_0^1 (t')^\alpha e^{-s'/t'}\left(\log\frac{1}{t'}\right)^{m-1} dt' \\
&\quad = \frac{1}{(m-1)!}\int_0^\infty |C_\mu(s'z)|(s')^{-\alpha-1}e^{-s'}ds' \int_0^1 (t')^\alpha \left(\log\frac{1}{t'}\right)^{m-1} dt' \\
&\quad \leqslant \text{const}\int_0^\infty |C_\mu(s'z)|(s')^{-\alpha-1}e^{-s'}ds' < \infty.
\end{aligned}$$

Thus $\mathcal{U}^m(\mu) \in \mathfrak{D}(\Phi_f)$ and

$$\int_0^\infty ds \int_0^1 C_\mu(u_m(t)f(s)z)dt = \int_0^1 dt \int_0^\infty C_\mu(u_m(t)f(s)z)ds,$$

which shows that $\Phi_f \mathcal{U}^m(\mu) = \mathcal{U}^m \Phi_f(\mu)$.

Step 5. Let m be a positive integer. Let $\mu \in I(\mathbb{R}^d)$ and $\widetilde{\mu} = \mathcal{U}^m(\mu)$. Then $\widetilde{\mu} \in \mathfrak{D}(\Phi_f)$ if and only if $\mu \in \mathfrak{D}(\Phi_f)$. The "if" part is already proved in Step 4, but the following proof shows it again. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$. Since

$$\widetilde{\nu}(B) = \int_0^1 ds \int_{\mathbb{R}^d} 1_B(u_m(s)x)\nu(dx) = \frac{1}{(m-1)!}\int_0^1 \left(\log\frac{1}{t}\right)^{m-1} dt \int_{\mathbb{R}^d} 1_B(tx)\nu(dx),$$

we have

$$\begin{aligned}
\int_{|x|>1} |x|^\alpha \widetilde{\nu}(dx) &= \frac{1}{(m-1)!}\int_0^1 \left(\log\frac{1}{t}\right)^{m-1} dt \int_{|tx|>1} t^\alpha |x|^\alpha \nu(dx) \\
&= \frac{1}{(m-1)!}\int_{|x|>1} |x|^\alpha \nu(dx) \int_{1/|x|}^1 \left(\log\frac{1}{t}\right)^{m-1} dt.
\end{aligned}$$

Since

$$\frac{1}{(m-1)!}\int_{1/|x|}^{1}\left(\log\frac{1}{t}\right)^{m-1}dt\to 1\quad\text{as } |x|\to\infty,$$

we see that $\int_{|x|>1}|x|^{\alpha}\widetilde{\nu}(dx)<\infty$ if and only if $\int_{|x|>1}|x|^{\alpha}\nu(dx)<\infty$. Now use the description of $\mathfrak{D}(\Phi_f)$ in Theorem B.

Step 6. We prove that

$$\mathfrak{R}_f^m\subset\mathcal{U}^m(I(\mathbb{R}^d))=U_{m-1}(\mathbb{R}^d),\qquad m=1,2,\ldots. \tag{8}$$

For $m=1$ this is already shown in Step 2. Suppose that $\mathfrak{R}_f^m\subset\mathcal{U}^m(I(\mathbb{R}^d))$. Let $\mu\in\Phi_f^m(\mathfrak{D}(\Phi_f^{m+1})$. Then $\mu\in\mathfrak{D}(\Phi_f)$ and $\mu=\mathcal{U}^m(\rho)$ for some $\rho\in I(\mathbb{R}^d)$, from which we have $\rho\in\mathfrak{D}(\Phi_f)$ using Step 5. Thus $\Phi_f(\mu)=\mathcal{U}^m\Phi_f(\rho)$ by Step 4. It follows that $\Phi_f^{m+1}(\mathfrak{D}(\Phi_f^{m+1}))\subset\mathcal{U}^m(\mathfrak{R}(\Phi_f))$, hence $\Phi_f^{m+1}(\mathfrak{D}(\Phi_f^{m+1}))\subset\mathcal{U}^{m+1}(I(\mathbb{R}^d))$.

Step 7. Let $\mu\in L_\infty(\mathbb{R}^d)$. Then $\mu\in\mathfrak{D}(\Phi_f)$ if and only if

$$\Gamma((0,\alpha])=0\quad\text{and}\quad\int_{(\alpha,2)}(\beta-\alpha)^{-1}\Gamma(d\beta)<\infty, \tag{9}$$

where Γ is the Γ-measure of μ To show this, use Theorem B and, in the notation of Theorem A, note that

$$\int_{|x|>1}|x|^{\alpha}\nu(dx)=\int_{(0,2)}\Gamma(d\beta)\int_S\lambda_\beta(d\xi)\int_1^\infty r^{\alpha-\beta-1}dr=\infty\quad\text{if }\Gamma((0,\alpha])>0,$$

and that, if $\Gamma((0,\alpha])=0$, then

$$\int_{|x|>1}|x|^{\alpha}\nu(dx)=\int_{(\alpha,2)}\Gamma(d\beta)\int_S\lambda_\beta(d\xi)\int_1^\infty r^{\alpha-\beta-1}dr=\int_{(\alpha,2)}(\beta-\alpha)^{-1}\Gamma^{\mu}(d\beta).$$

Step 8. If $\mu\in L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f)$, then $\Phi_f(\mu)\in L_\infty(\mathbb{R}^d)$ and the Lévy measure $\widetilde{\nu}$ of $\Phi_f(\mu)$ is as follows:

$$\widetilde{\nu}(B)=\int_{(\alpha,2)}\Gamma(\beta-\alpha)\Gamma(d\beta)\int_S\lambda_\beta(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr,\qquad B\in\mathcal{B}(\mathbb{R}^d), \tag{10}$$

where $\Gamma(d\beta)$ and $\lambda_\beta(d\xi)$ are those of μ in Theorem A. Recall that $\Gamma(\beta-\alpha)\sim(\beta-\alpha)^{-1}$ as $\beta\downarrow\alpha$, since $(\beta-\alpha)\Gamma(\beta-\alpha)=\Gamma(\beta-\alpha+1)\to\Gamma(1)=1$ as $\beta\downarrow\alpha$. Indeed,

$$\begin{aligned}
\widetilde{\nu}(B)&=\int_0^\infty ds\int_{\mathbb{R}^d}1_B(f(s)x)\nu(dx)\\
&=\int_0^\infty p(u)du\int_{\mathbb{R}^d}1_B(ux)\nu(dx)\\
&=\int_0^\infty u^{-\alpha-1}e^{-u}du\int_{(\alpha,2)}\Gamma(d\beta)\int_S\lambda_\beta(d\xi)\int_0^\infty 1_B(ur\xi)r^{-\beta-1}dr\\
&=\int_{(\alpha,2)}\Gamma(d\beta)\int_S\lambda_\beta(d\xi)\int_0^\infty u^{-\alpha-1}e^{-u}du\int_0^\infty 1_B(r'\xi)u^{\beta}(r')^{-\beta-1}dr'
\end{aligned}$$

$$= \int_{(\alpha,2)} \Gamma(\beta-\alpha)\Gamma(d\beta)\int_S \lambda_\beta(d\xi)\int_0^\infty 1_B(r'\xi)(r')^{-\beta-1}dr'.$$

Step 9. Let us show that

$$\Phi_f(L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f)) = L_\infty^{(\alpha)}(\mathbb{R}^d). \tag{11}$$

It follows from Step 8 that the left-hand side is included in the right-hand side. Let $\mu = \mu_{(A,\nu,\gamma)} \in L_\infty^{(\alpha)}(\mathbb{R}^d)$ with ν represented by $\Gamma(d\beta)$ and $\lambda_\beta(d\xi)$. Let $\mu_0 \in I(\mathbb{R}^d)$ with triplet (A_0,ν_0,γ_0) defined by

$$A_0 = (\Gamma(2-\alpha))^{-1}A,$$
$$\nu_0(B) = \int_{(\alpha,2)} (\Gamma(\beta-\alpha))^{-1}\Gamma(d\beta)\int_S \lambda_\beta(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr,$$
$$\gamma_0 = (\Gamma(1-\alpha))^{-1}(\gamma - c_{\nu_0}),$$

where

$$c_{\nu_0} = \int_0^\infty f(s)ds\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu_0(dx).$$

Definability of c_{ν_0} is checked in the following way:

$$\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu_0(dx) = \int_{\mathbb{R}^d}\frac{x(|x|^2-|f(s)x|^2)\nu_0(dx)}{(1+|f(s)x|^2)(1+|x|^2)},$$
$$\int_0^\infty f(s)ds\int_{\mathbb{R}^d}\frac{|x|^3\nu_0(dx)}{(1+|f(s)x|^2)(1+|x|^2)} \leqslant \frac{1}{2}\int_{\mathbb{R}^d}\frac{|x|^2\nu_0(dx)}{1+|x|^2} < \infty,$$
$$\int_0^\infty f(s)ds\int_{\mathbb{R}^d}\frac{|x|\,|f(s)x|^2\nu_0(dx)}{(1+|f(s)x|^2)(1+|x|^2)} = \int_0^\infty f(s)^2ds\int_{\mathbb{R}^d}\frac{|x|^2|f(s)x|\nu_0(dx)}{(1+|f(s)x|^2)(1+|x|^2)}$$
$$\leqslant \frac{\Gamma(2-\alpha)}{2}\int_{\mathbb{R}^d}\frac{|x|^2\nu_0(dx)}{1+|x|^2} < \infty,$$

since $\int_0^\infty f(s)^2ds = \int_0^\infty u^2p(u)du = \int_0^\infty u^{-\alpha+1}e^{-u}du = \Gamma(2-\alpha)$. Thus $\mu_0 \in L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f)$ by Step 7. Using Step 8 and Proposition 2.6 of [S06], we see that $\Phi_f(\mu_0) = \mu$, since

$$\int_0^\infty f(s)^2A_0ds = \Gamma(2-\alpha)A_0 = A,$$
$$\lim_{t\to\infty}\int_0^t f(s)ds\left(\gamma_0 + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu_0(dx)\right)$$
$$= \Gamma(1-\alpha)\gamma_0 + c_{\nu_0} = \gamma,$$

noting that $\int_0^\infty f(s)ds = \int_0^\infty up(u)du = \int_0^\infty u^{-\alpha}e^{-u}du = \Gamma(1-\alpha)$. This proves that $L_\infty^{(\alpha)}(\mathbb{R}^d) \subset \Phi_f(L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f))$.

Step 10. We claim that, for any positive integer m,

$$\Phi_f^m(L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f^m)) = L_\infty^{(\alpha)}(\mathbb{R}^d). \tag{12}$$

It follows from Step 9 that

$$\Phi_f^m(L_\infty \cap \mathfrak{D}(\Phi_f^m)) \subset \Phi_f(L_\infty \cap \mathfrak{D}(\Phi_f)) = L_\infty^{(\alpha)}.$$

Let us show

$$\Phi_f^m(L_\infty \cap \mathfrak{D}(\Phi_f^m)) \supset L_\infty^{(\alpha)} \tag{13}$$

by induction. This is true for $m = 1$ from Step 9. Suppose that (13) is true for a given m. Then, using Step 9,

$$\begin{aligned} L_\infty^{(\alpha)} &\subset \Phi_f^m(L_\infty^{(\alpha)} \cap \mathfrak{D}(\Phi_f^m)) = \Phi_f^m(\Phi_f(L_\infty \cap \mathfrak{D}(\Phi_f)) \cap \mathfrak{D}(\Phi_f^m)) \\ &= \Phi_f^m(\Phi_f(L_\infty \cap \mathfrak{D}(\Phi_f^{m+1}))) = \Phi_f^{m+1}(L_\infty \cap \mathfrak{D}(\Phi_f^{m+1})). \end{aligned}$$

Hence (13) is true for all m.

Final step. It follows from Step 10 that $\bigcap_{m=1}^\infty \mathfrak{R}_f^m \supset L_\infty^{(\alpha)}(\mathbb{R}^d)$. Let us show the converse inclusion. It follows from Step 6 that

$$\bigcap_{m=1}^\infty \mathfrak{R}_f^m \subset \bigcap_{m=1}^\infty U_{m-1}(\mathbb{R}^d) = U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d).$$

Here we have used Jurek's result that $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. Next, we claim that if $\mu \in L_\infty(\mathbb{R}^d) \cap \mathfrak{R}_f^1$, then $\Gamma^\mu((0,\alpha]) = 0$. Indeed, if $\mu \in \mathfrak{R}_f^1$, then the Lévy measure ν^μ has expression using $\lambda(d\xi)$ and $h_\xi(u)$ in Theorem B. On the other hand, if $\mu \in L_\infty(\mathbb{R}^d)$, then ν^μ has expression using $\Gamma(d\beta) = \Gamma^\mu(d\beta)$ and $\lambda_\beta(d\xi)$ in Theorem A, which is rewritten as

$$\begin{aligned} \nu^\mu(B) &= \int_S \overline{\lambda}(d\xi) \int_{(0,2)} \Gamma_\xi(d\beta) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr \\ &= \int_S \overline{\lambda}(d\xi) \int_0^\infty 1_B(r\xi) dr \int_{(0,2)} r^{-\beta-1} \Gamma_\xi(d\beta), \end{aligned}$$

where $\overline{\lambda}(d\beta)$ is a probability measure on S and $\Gamma_\xi(d\beta)$ is, for each $\xi \in S$, a measure on $(0,2)$ such that $\int_{(0,2)}(\beta^{-1} + (2-\beta)^{-1})\Gamma_\xi(d\beta) = \text{const}$ and Γ_ξ is measurable in ξ. In fact, $\Gamma(d\beta)\lambda_\beta(d\xi) = \overline{\lambda}(d\xi)\Gamma_\xi(d\beta)$. Now use the uniqueness of the polar decomposition in Lemma 2.1 of [BMS06]. Thus, if $\mu \in L_\infty(\mathbb{R}^d) \cap \mathfrak{R}_f^1$, then there is a positive finite measurable function $c(\xi)$ such that $\lambda(d\xi) = c(\xi)\overline{\lambda}(d\xi)$ and that, for λ-a.e. ξ,

$$h_\xi(r) = c(\xi)^{-1} r^{\alpha+1} \int_{(0,2)} r^{-\beta-1} \Gamma_\xi(d\beta) = c(\xi)^{-1} \int_{(0,2)} r^{\alpha-\beta} \Gamma_\xi(d\beta).$$

Since $h_\xi(r) \to 0$ as $r \to \infty$, we obtain $\Gamma_\xi((0,\alpha]) = 0$, which implies $\Gamma((0,\alpha]) = 0$. This completes the proof that $\bigcap_{m=1}^\infty \mathfrak{R}_f^m = L_\infty^{(\alpha)}(\mathbb{R}^d)$.

Remarks. Let $f(s)$ and Φ_f be as in Theorems B and C. We show some properties of Φ_f which we did not use in the proof of Theorem C.

1. Let $\beta \in (0,\alpha]$. Then any non-trivial β-stable distribution μ does not belong to $\mathfrak{D}(\Phi_f)$, because $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) = \infty$.

2. If $\alpha < \beta < 2$, then $S_\beta(\mathbb{R}^d) \subset \mathfrak{D}(\Phi_f)$ and

$$\Phi_f(S_\beta(\mathbb{R}^d)) = S_\beta(\mathbb{R}^d),$$

where $S_\beta(\mathbb{R}^d)$ is the class of β-stable distributions on $\mathbb{R}^d$. Indeed, the first assertion comes from Theorem B and note that

$$\int_0^\infty f(s)ds = \int_0^\infty up(u)du = \int_0^\infty u^{-\alpha}e^{-u}du < \infty,$$

since $\alpha < 1$ and that

$$\int_0^\infty f(s)^\beta ds = \int_0^\infty u^\beta p(u)du = \int_0^\infty u^{\beta-\alpha-1}e^{-u}du < \infty,$$

since $\alpha < \beta$. Thus the proof of Lemma 3.7 of [MS07] works.

3. Let m be a positive integer. Let $\mu \in L_\infty(\mathbb{R}^d)$ with Lévy measure represented by Γ and λ_β in Theorem A. Then $\mu \in \mathfrak{D}(\Phi_f^m)$ if and only if

$$\Gamma((0,\alpha]) = 0 \quad \text{and} \quad \int_{(\alpha,2)} (\beta-\alpha)^{-m}\Gamma(d\beta) < \infty.$$

If $\mu \in \mathfrak{D}(\Phi_f^m)$, then $\Phi_f^m(\mu) \in L_\infty(\mathbb{R}^d)$ and the Lévy measure $\widetilde{\nu}$ of $\Phi_f^m(\mu)$ is as follows:

$$\widetilde{\nu}(B) = \int_{(\alpha,2)} (\Gamma(\beta-\alpha))^m \Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1}dr, \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Further,

$$\Phi_f^m(L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f^m)) = L_\infty^{(\alpha)}(\mathbb{R}^d).$$

To prove this, repeat the arguments in Steps 7, 8, and 9.

4. Let m be a positive integer. A distribution $\mu \in I(\mathbb{R}^d)$ is in $\mathfrak{D}(\Phi_f^m)$ if and only if it has Lévy measure ν satisfying

$$\int_{|x|>1} |x|^\alpha (\log|x|)^{m-1}\nu(dx) < \infty. \tag{14}$$

To see this, first note that

$$\int_{1/v}^\infty u^{-1}e^{-u}(\log uv)^{m-1}du \sim m^{-1}(\log v)^m, \qquad v \to \infty. \tag{15}$$

Indeed, for $m = 1$, this is seen using l'Hopital's rule; if (15) is true for m, then

$$\lim_{v\to\infty} \frac{\int_{1/v}^\infty u^{-1}e^{-u}(\log uv)^m du}{(m+1)^{-1}(\log v)^{m+1}} = \lim_{v\to\infty} \frac{m\int_{1/v}^\infty u^{-1}e^{-u}(\log uv)^{m-1}v^{-1}du}{(\log v)^m v^{-1}} = 1$$

using l'Hopital's rule again. Now, the assertion is true for $m = 1$ as is in Theorem B. Suppose that the assertion is true for a given m. Then $\mathfrak{D}(\Phi_f^{m+1})$ is the class of $\mu \in \mathfrak{D}(\Phi_f)$ such that $\Phi_f(\mu)$ has Lévy measure $\widetilde{\nu}$ satisfying $\int_{|x|>1} |x|^\alpha (\log |x|)^{m-1} \widetilde{\nu}(dx) < \infty$. But

$$\int_{|x|>1} |x|^\alpha (\log |x|)^{m-1} \widetilde{\nu}(dx) = \int_0^\infty p(u)du \int_{|ux|>1} |ux|^\alpha (\log |ux|)^{m-1} \nu(dx)$$

$$= \int_{|x|>0} |x|^\alpha \nu(dx) \int_{1/|x|}^\infty u^{-1} e^{-u} (\log |ux|)^{m-1} du,$$

$$\int_{0<|x|\leqslant 1} |x|^\alpha \nu(dx) \int_{1/|x|}^\infty u^{-1} e^{-u} (\log |ux|)^{m-1} du$$

$$\leqslant \int_{0<|x|\leqslant 1} |x|^\alpha \nu(dx) \int_{1/|x|}^\infty u^{-1} e^{-u} (\log u)^{m-1} du < \infty,$$

since

$$\int_v^\infty u^{-1} e^{-u} (\log u)^{m-1} du \sim v^{-1} e^{-v} (\log v)^{m-1}, \qquad v \to \infty.$$

We have

$$\int_{|x|>1} |x|^\alpha \nu(dx) \int_{1/|x|}^\infty u^{-1} e^{-u} (\log |ux|)^{m-1} du < \infty$$

if and only if $\int_{|x|>1} |x|^\alpha (\log |x|)^m \nu(dx) < \infty$ by virtue of (15). Hence the assertion is true for $m + 1$.

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Correction. Page 1, the first line of the text: replace "classes" by "sequence".