Banach lattices, potential operators, population-genetic models, L distributions, and Lévy processes

Ken-iti Sato

Note. On June 25, 1994, Extended Nagoya Probability Seminar was organized for the occasion of my *kanreki*[1]. Three talks were given:
Minoru Motoo, Mr. Ken-iti Sato around 1960-1970,
Makoto Yamazato, In the joint work with Sato-san,
Ken-iti Sato, Banach lattices, potential operators, population-genetic models, L distributions, and Lévy processes.
At that time I wrote and privately distributed a report of my talk in Japanese. The present article is its translation by myself written up in February 2020. Corrected or inserted sentences are given in braces { }.

Part I is an extension of my talk on June 25, 1994. Part II is a slight enlargement of the list that was handed out. {The items in Part II are cited by the numbers there such as [1.1], [2.1], and *[2.5]. }

Part I

College of General Education of Nagoya University was reorganized to Faculty and Graduate School of Informatics and Sciences, after some exchange of members with other faculties. In this process we had to write up our lists of papers to a review committee repeatedly with explanation for each paper. Looking back to my papers brought various thoughts to me. Paul Lévy recollected his works in his autobiography *Quelques Aspects de la Pensée d'un Mathématicien*, Albert Blanchard, 1970 (Japanese translation by T. Hida and K. Yamamoto, Iwanami, 1973), making various comments, sometimes being regretful. I have no work with a scale as big as his, but I am also going to talk some thoughts on my papers.

[1]{The sixtieth birthday is called *kanreki*, which means return to the same year according to the traditional Chinese calendar imported to Japan. In the calendar each of 60 years has a different name so that the name of year is periodic with period 60. Actually two sets of periodic names are combined, one has period 10 (5 natural elements, each elder and younger), the other has period 12 (animal names), and 60 is the least common multiple of 10 and 12.}

1. Boundary behavior problem of Markov processes.

Let me add something to Motoo's talk today. In 1966-1967 I was working hard for analytic construction of multi-dimensional diffusions satisfying A.D. Wentzell's boundary condition[2]. I reported on this subject at meetings of Math. Society of Japan and seminars at University of Minnesota, which were *[2.5], *[2.6], and *[2.7]. Among them *[2.5] was a joint work with Motoo. To our regret, we did not write those results as a paper. It was because J.-M. Bony, Ph. Courrège, and P. Priouret in France continued our works (especially [2.3] of T. Ueno and me) and wrote papers first in C. R. Acad. Sci. Paris Ser. A-B 263 (1966), A451–A454, and then Séminaire Brelot–Choquet–Deny, 10, fasc. 1 et 2, (1965/66) and Ann. Inst. Fourier (Grenoble), tom. 18, fasc. 2 (1969), 369–521, which made our writing a hard task. However the method of those joint authors in France was different from ours so that the obtained sufficient conditions for the construction were not the same. So our results would have had some value, but we wanted to improve it and eventually did not write it. For me the boundary behavior problem was thus left unfinished.

Among Japanese probabilists, S. Watanabe[3] showed pathwise construction of diffusions with Wentzell's boundary condition and Motoo[4] gave beautiful results on probabilistic decomposition of excursions from and to boundary. {R.M. Blumenthal's book[5] *Excursions of Markov Processes*, Birkhäuser, 1992, dedicated one chapter to excursions away from a set, describing Motoo's paper as one of the most important in the field.} However, detailed proofs of Motoo's results of his 1967 paper are available only in Japanese[6].

Unfortunately, we did not discover Wentzell's boundary condition in Japan. Several of us had studied one-dimensional works of Feller, Dynkin, and Ito–McKean and were thinking about multi-dimensional problems. If they had set up the problem and tackled it, they should have found the form of the boundary conditions without difficulty. But it is questionable that they could have made explicit construction of the solution in the case of the ball with the boundary condition having constant

[2]Teor. Veroyatn. Primen., 4 (1959), 172–185.

[3]Banach Center Publ., Vol. 5 (1979), 255–271.

[4]{Proc. Fifth Berkeley Symp. Math. Statist. Probab., Vol. 2, Part 2, 1967, 75–110.}

[5]{I did not know the book in 1994.}

[6]{H. Kunita, K. Sato, M. Fukushima, and M. Motoo, Diffusion processes and Markov processes on boundary (in Japanese), Seminar on Probability, Vol. 22, 161 pages, published by Kakuritsuron Seminar in 1965.}

coefficients by the use of spherical functions, as Wentzell's paper did. Our basis in analysis was weaker.

Theory of Markov processes in Japan revolved on the axis of boundary behavior problems and extended to larger subjects. The history should be written, reflecting on the works of Motoo, N. Ikeda, Takesi Watanabe, T. Ueno, H. Tanaka, S. Watanabe, M. Nagasawa, M. Fukushima, H. Kunita, Y. Okabe, and others.

2. Banach latices.

The real Banach spaces C, L^p, C^* etc. have an order structure. A Banach lattice is their generalization; it is a real Banach space and, at the same time, a lattice with semi-order $f \leqslant g$ (that is, for every f and g, $f \vee g = \sup\{f, g\}$ and $f \wedge g = \inf\{f, g\}$ are defined and hence $|f| = f \vee (-f)$, $f^+ = f \vee 0$, and $f^- = -(f \wedge 0)$ are also defined) and moreover it satisfies

$$
\begin{gathered}
f \leqslant g \quad \Longrightarrow \quad f + h \leqslant g + h, \\
f \leqslant g, \;\; a \in \mathbb{R}_+ \quad \Longrightarrow \quad af \leqslant ag, \\
f \leqslant g \quad \Longrightarrow \quad -f \geqslant -g, \\
|f| \leqslant |g| \quad \Longrightarrow \quad \|f\| \leqslant \|g\|.
\end{gathered}
$$

My work in Banach lattices was no more than the discovery of good functional $\sigma(f, g)$; I remember that I was very glad when I found it. Minoru Hasegawa[7] was the first who used

$$\tau(f, g) = \lim_{\epsilon \downarrow 0} \epsilon^{-1}(\|f + \epsilon g\| - \|f\|)$$

in the theory of operator semigroups. In [4.1] I introduced for $f \geqslant 0$

$$\sigma(f, g) = \inf_k \lim_{b \to \infty} \tau(f, (g + k) \vee (-bf)),$$

where k runs over all elements satisfying $f \wedge |k| = 0$. This functional $\sigma(f, g)$ has nice properties and is useful in characterization of the infinitesimal generator of a positive contraction semigroup; it is usable further in the theory of sums of infinitesimal generators. The following are some of its nice properties:

$$
\begin{gathered}
-\|g^-\| \leqslant \sigma(f, g) \leqslant \|g^+\|, \\
\sigma(f, g + h) \leqslant \sigma(f, g) + \sigma(f, h), \\
g \leqslant h \quad \Longrightarrow \quad \sigma(f, g) \leqslant \sigma(f, h), \\
f \wedge |h| = 0 \quad \Longrightarrow \quad \sigma(f, g) = \sigma(f, g + h).
\end{gathered}
$$

[7] J. Math. Soc. Japan, 18 (1966), 290–302.

Let A be the Hille–Yosida infinitesimal generator of a strongly continuous semigroup T_t with domain $\mathfrak{D}(A)$. Then T_t is a positive contraction semigroup if and only if

$$\sigma(f^+, Af) \leqslant 0 \quad \text{for all } f \in \mathfrak{D}(A). \tag{1}$$

I called the property (1) *dispersive*. This can be used also for nonlinear operators. The concrete representations of $\sigma(f,g)$ are, for $f \leqslant 0$ with $f \neq 0$,

$$\sigma(f,g) = \max_{f(x)=\|f\|} g(x) \text{ in } C(X) \text{ or } C_0(X),$$
$$\sigma(f,g) = \int_{\{f(x)>0\}} g(x)m(dx) \text{ in } L^1(X,m),$$
$$\sigma(f,g) = \|f\|^{-p+1} \int_X f(x)^{p-1} g(x)m(dx) \text{ in } L^p(X,m),\ 1 < p < \infty,$$
$$\sigma(f,g) = g_f^{\mathrm{ac}}(X) \text{ in } C^*(X) \text{ where } g_f^{\mathrm{ac}} \text{ is the absolutely continuous part of } g \text{ with respect to } f,$$

and so on. Hence, for $C(X)$, the property (1) means that $Af(x) \leqslant 0$ at the point x where $f(x) = \max_y f(y) > 0$, that is, the characterization of elliptic operator.

A Markov semigroup T_t in the space L^2 further satisfies $T_t f \leqslant 1$ for $f \leqslant 1$; the characterization of its infinitesimal generator was given by Kunita (Proc. Internat. Conf. on Func. Anal. and Rel. Topics, Tokyo, 1970, 332–343). I gave its generalization in [4.4]. There I showed that

$$\rho(f,g) = \lim_{\epsilon \downarrow 0} \epsilon^{-1}(\|(f+\epsilon g)^+\| - \|f^+\|)$$

can also be used for the same purpose as $\sigma(f,g)$, but $\sigma(f,g)$ has nicer forms in concrete Banach lattices.

The functional $\sigma(f,g)$ was employed by Yoshio Konishi (Proc. Japan Acad., 47 (1971), 24–28; 48 (1972), 281–286, etc.) for nonlinear evolution equations. But I am disappointed that books on Banach lattices or positive semigroups do not use this functional even when its use could make their argument simpler. (After this talk Hideo Nagai told me that L.C. Evans and his collaborator[8] refer to [4.1], but its full use is not done.)

3. Potential operators.

When I was in the fourth year of the undergraduate and in the master course in University of Tokyo, my adviser was K. Yosida (1909–1990). The Hille–Yosida

[8] L.C. Evans, Indiana Univ. Math. J., 27 (1978), 875–887; L.C. Evans and A. Friedman, Trans. A. M. S., 253 (1979), 365–389; L.C. Evans, Israel J. Math., 36 (1980), 225–247.

theory of semigroups of linear operators is famous; this work was done independently by the two (Yosida's was J. Math. Soc. Japan, 1 (1948), 15–21). Twenty years after, Yosida wrote a paper that is a direct continuation (Studia Math., 31 (1968), 531–533). Namely, let T_t be a strongly continuous semigroup of linear operators in a Banach space $\mathfrak{B}$, A its infinitesimal generator, and J_λ the resolvent of A given by

$$J_\lambda f = (\lambda - A)^{-1} f = \int_0^\infty e^{-\lambda t} T_t f dt \quad \text{for } \lambda > 0.$$

Then he said that T_t has potential operator V if $\operatorname*{s-lim}_{\lambda\downarrow 0} J_\lambda f$ exists for dense f in $\mathfrak{B}$, and defined Vf by this limit. Next he showed that each of the following three statements is a necessary and sufficient condition for T_t to have potential operator:

$$\text{the range } \mathfrak{R}(A) \text{ of } A \text{ is dense in } \mathfrak{B}, \tag{2}$$

$$\operatorname*{s-lim}_{\lambda\downarrow 0} \lambda J_\lambda f = 0 \text{ for all } f, \tag{3}$$

$$\operatorname*{s-lim}_{t\uparrow\infty} t^{-1} \int_0^t T_s f ds = 0 \text{ for all } f. \tag{4}$$

If T_t has potential operator V, then A is one-to-one, $\mathfrak{D}(V) = \mathfrak{R}(A)$, and $V = A^{-1}$. Sometimes V is called the potential operator in Yosida's sense.

I showed in [5.1] that if the Markov process associated with a semigroup T_t in C_0 is transient or null-recurrent, then T_t has potential operator, and that if the process is positive-recurrent, then T_t does not have potential operator. I also calculated the potential operators in the case of stable processes and some others..

I would like to mention that, in [5.2], I pointed out that F. Hirsch's paper in 1970 (C. R. Acad. Sci. Paris, 270, 1487–1490) and Yosida's paper in 1972 (Publ. R. I. M. S., 8, 201–205) say essentially the following result on Hilbert spaces and that it cannot be extended to Banach spaces.

If $\mathfrak{B}$ is a Hilbert space and A is a linear operator with domain dense in $\mathfrak{B}$, then the following two conditions (5) and (6) are equivalent:

(5) A is the infinitesimal generator of a strongly continuous contraction semigroup $T_t^{(1)}$ with potential operator,

(6) $-A$ is the potential operator of a strongly continuous contraction semigroup $T_t^{(2)}$.

I wonder how $T_t^{(1)}$ and $T_t^{(2)}$ are intrinsically connected. They are adjoints in some sense. In the case of $\mathfrak{B} = L^2$, even if one of $T_t^{(1)}$ and $T_t^{(2)}$ is positive, the other is not necessarily positive. If $\mathfrak{B}$ is a Banach space, then, under the condition (5), the condition (6) is equivalent to

$$\|\lambda J_\lambda - 1\| \leqslant 1 \quad \text{for all } \lambda > 0.$$

If $\mathfrak{B}$ is a Hilbert space, then always $\|\lambda J_\lambda - 1/2\| \leqslant 1/2$.

4. Tails of infinitely divisible distributions.

The paper [6.1] published in 1973 is my first work on infinitely divisible distributions themselves, although infinitesimal generators and potential operators of Lévy processes were treated before that. Let me state main results in one dimension (d-dimensional case is similar). Let μ be an infinitely divisible distribution with Lévy measure ν.

(i) If $g(x)$ is submultiplicative, then $\int g(x)\mu(dx) < \infty$ and $\int_{|x|>1} g(x)\nu(dx) < \infty$ are equivalent. Here we call $g(x)$ submultiplicative if $g(x) \geqslant 0$ and there is $a > 0$ such that $g(x+y) \leqslant a g(x) g(y)$. For example, if $\alpha > 0$, then $(1 \vee \log|x|)^\alpha$, $1 \vee |x|^\alpha$, and $e^{\alpha|x|}$ are submultiplicative.

(ii) Let $b = \inf\{r : \nu(|x| > r) = 0\}$ with convention that $\inf \varnothing = \infty$. Then $\int e^{\alpha|x|\log|x|}\mu(dx)$ is finite for $\alpha < 1/b$ and infinite for $\alpha > 1/b$.

Assertion (ii) or its expression in the order of decrease of the tail is fairly often cited among my results. First I submitted a paper containing only (i) to Ann. Math. Stat. and it was accepted {with a minor revision assumed}. Just at that time it was announced that Ann. Math. Stat. would be divided into Ann. Prob. and Ann. Stat. So I told that I preferred Ann. Prob. Then the paper was refereed again, and rejected by Ann. Prob. by reason that the same result existed in V.M. Kruglov's paper (Teor. Veroyatn. Primen., 15 (1970), 330–336). Thus I found the result (i) was already given by Kruglov in one dimension. Kruglov also showed (i) in d dimensions and Hilbert spaces later. Further he showed in one dimension that boundedness of the support of ν implies the finiteness of the integral in (ii) for some $\alpha > 0$, which is a part of (ii). So I thoroughly generalized it in the form (ii) in d dimensions, where V.M. Zolotarev's estimate[9] of large deviations was helpful.

The paper [6.1] does not seem to have attracted the interest of many people, but I was encouraged by Gisiro Maruyama (1916–1986), who wrote the paper[10] on infinitely divisible processes shortly before. I feel that I got some intuition in the relation of infinitely divisible distributions and their Lévy measures around the time

[9]Teor. Veroyatn. Primen., 10 (1965), 33–50.

[10]Teor. Veroyatn. Primen., 15 (1970), 3–23.

when I wrote [6.1]. However, the relation is quite complicated; the problem of time evolution of the distribution of a Lévy process is connected with this relation and many problems are unsolved even in one dimension.

5. Population-genetic models.

Main works in this subject are [7.1], [8.1], [8.2], and [8.6]; except [8.6], they were done 1975–1975. In 1964[11] and 1965[12] S. Karlin and J. McGregor introduced as an object of research the following Markov chains induced by direct product branching processes. They are treated also in Karlin's book (A First Course in Stochastic Processes, Academic Press, 1969)[13]. Let $Z_1(n)$ and $Z_2(n)$ be independent Galton–Watson processes (branching processes) with a common distribution of the number of offspring of one individual. Let $f(s) = \sum_{k=0}^{\infty} c_k s^k$ be the generating function of the distribution and assume that $c_0 c_1 c_2 > 0$. For any positive integer N, let

$$P_{jk}^{(N)} = P(Z_1(n+1) = k \mid Z_1(n) = j, Z_2(n) = N - j, Z_1(n+1) + Z_2(n+1) = N)$$

and consider the Markov chain on the set $\{0, 1, \ldots, N\}$ with one-step transition probability $P_{jk}^{(N)}$. They showed that, if they choose $f(s)$ appropriately, sometimes making a modification of this model, many a population-genetic model is realized as this Markov chain. Further they showed that the transition matrix $(P_{jk}^{(N)})$ is diagonalizable with eigenvalues

$$1 = \lambda_0^{(N)} = \lambda_1^{(N)} > \lambda_2^{(N)} > \cdots > \lambda_N^{(N)} > 0$$

and proved that

$$\lambda_r^{(N)} = \frac{\text{coefficient of } s^{N-r} \text{ in } f(s)^{N-r} f'(s)^r}{\text{coefficient of } s^N \text{ in } f(s)^N} \tag{7}$$

for $r = 0, 1, \ldots, N$. This is the model of two alleles, but the case of three or more alleles can be treated similarly and the step of mutation and immigration can be inserted. But natural selection cannot be taken into account. An especially important quantity is $\lambda_2^{(N)}$, which represents the speed of approach to fixation. They showed that if $f(s)$ is of Poisson, binomial, or negative-binomial, then

$$1 - \lambda_2^{(N)} \sim \frac{\text{const}}{N} \quad \text{as } N \to \infty. \tag{8}$$

[11]Proc. Nat. Acad. Sci. U. S. A., 51, 598–602.

[12]Bernoulli Bayes Laplace Anniversary Volume, Springer, 111–145.

[13]It is translated into Japanese by Ken-iti Sato and Yumiko Sato (Kakuritsu Katei Kôgi, Sangyô Tosho, 1974). {My first encounter with this book was as textbook when I taught an undergraduate course on stochastic processes at University of Illinois in 1968/69.}

W.J. Ewens[14] remarked that if the distribution for $f(s)$ has mean 1, then the constant in (8) is its variance σ^2. I was interested whether (8) is true in general, what is the meaning of the constant if (8) is true, what can be said about $\lambda_r^{(N)}$, what happens when both r and N go to infinity, and so on. This was my motivation for studying genetic models. I hardly had personal influence of anybody in Japan, {although there was a strong group of population geneticists headed by M. Kimura}[15]. However, genetic models lay as one of the starting points of W. Feller's study of diffusion processes[16]. As early as 1922 R.A. Fisher mentioned the heat equation for the temporal change of gene frequency.

Looking at the formula (7), I thought the analysis of large deviations must be helpful in order to study behavior of $\lambda_r^{(N)}$ as $N \to \infty$. Thus I studied the works of H. Cramér, V.V. Petrov, W. Richter, I.A. Ibragimov–Yu.V. Linnik, etc. Among them I learned Cramér[17] intensively and was impressed by his argument (Laplace method). Other people's works were development of Cramér's. In the book by Ibragimov and Linnik (English translation is Independent and stationary sequences of random variables, Wolters-Noordhoff, 1971), Linnik writes the theory of large deviations under conditions weaker than Cramér's, but it was so difficult that I could not check a number of places. Soon after the appearance of the three papers of M.D. Donsker and S.R.S. Varadhan (Comm. Pure Appl. Math., 28), the theory became popular also in Japan but, earlier, I was only one tackling large deviations theory in the probability group in Japan, I think.

Using the method of large deviations, I obtained in [7.1]

$$(9) \qquad 1 - \lambda_r^{(N)} = \frac{a_{r,1}}{N} + \frac{a_{r,2}}{N^2} + \cdots + \frac{a_{r,p}}{N^p} + O\left(\frac{1}{N^{p+1}}\right) \quad \text{as } N \to \infty$$

and expression of the coefficients ($a_{r,1} = \sigma^2 r(r-1)/2$ and so on). The case where r and N get bigger simultaneously could also be treated; for example, if $c > 0$ is fixed, then

$$(10) \qquad \lambda_{[c\sqrt{N}]}^{(N)} = e^{-\sigma^2 c^2/2}\left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \quad \text{as } N \to \infty.$$

[14]Population genetics, Mathuen, 1969. See page 41.

[15]{A bit later A. Shimizu and T. Shiga began to write on genetic models.}

[16]Proc. Second Berkeley Symp. Math. Stat. Prob., 1951, 227–246; Ann. Math., 54 (1951), 173–182.

[17]Actualités Scientifiques et Industrielles, No. 736, 1938, 5–23.

This Markov chain converges to the diffusion process determined by

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} x(1-x) \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 \leqslant x \leqslant 1$$

as $N \to \infty$, if we make a suitable change of time and space scales. As the differential operator in the right-hand side has discrete spectrum λ_r, $r = 0, 1, \dots$ and $\lambda_r = a_{r,1}$, the first term in (9) expresses the convergence of spectrum

$$(\lambda_r^{(N)})^{Nt} \to e^{-a_{r,1}t} \quad \text{as } N \to \infty.$$

The remaining terms in (9) would be related to the speed of the convergence of this Markov chain to the diffusion process. Further, the eigenvectors would converge to the eigenfunctions. But I did not go to proving them. Conversely, for a like model, Karlin and McGregor[18] demonstrated the convergence of a Markov chain to a diffusion from the convergence of eigenvalues and eigenvectors.

The Markov chain of Karlin and McGregor converges to the diffusion process

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} x \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 \leq x < \infty$$

as $N \to \infty$, if we make some other change of scales; the differential operator in the right-hand side has a continuous spectrum except a point spectrum at 0. The result (10) would show the convergence of spectral density in this case, but I did not prove it. The eigenvector might converge to the generalized eigenfunction.

In the paper [8.1] I treated the convergence to a multi-dimensional diffusion process for a generalization of the above Markov chain. However, in the case of the dimension $\geqslant 2$, I could not prove the uniqueness of the corresponding diffusion process except in some special cases, because of the degeneracy on the boundary of the limit diffusion operator. While I was preparing the paper [8.1], in November 1975 at Chicago, I was introduced to S.N. Ethier and learned that he had worked on the same equation in his doctor thesis and found a nice method to prove the uniqueness, employing that the coefficients of the second order derivatives are polynomials of order 2. Ethier proposed me to unite his result and my yet unpublished [8.1] to a joint paper; it would have been a very good paper. But I did not agree, as I thought that it would be better for a first paper of a young student to be written only by himself. His paper appeared in Comm. Pure Appl. Math., 29 (1976), 483-493.

I succeeded to deal with the convergence of genetic models involving natural selection to diffusion processes in [8.2]. Thus I came out of the Karlin–McGregor

[18] Proc. Camb. Phil. Soc., 58 (1962), 299–311.

model for the first time. I made asymptotic estimate of moments, fully using the Laplace method.

Let us proceed to [8.6]. {We consider a Markov chain on the d-dimensional (d-allelic) nonnegative lattice points with the sum of components being N, where one-step transition consists of two stages — independent reproduction and sampling of size N. If the distributions of offspring numbers have a common mean for all alleles, then the difference of variances affects the form of the limiting $(d-1)$-dimensional diffusion operator[19] as $N \to \infty$, where the time span for one-step transition is equal to N^{-1}.} When $d = 2$, this fact was discovered by J.H. Gillespie (Genetics, 76 (1974), 601–606; 77 (1975), 403–413) and he showed, with heuristic proofs, that the limiting diffusion operator has a polynomial of order 3 in the coefficient of the second-order derivative and the smaller variance works as advantageous selection force. M.F. Norman called my attention to Gillespie's paper. I was interested what form of coefficients would appear in the case $d \geqslant 3$ and how the convergence would be justified. This was done in [8.6]. If mutation is added in this model, then the boundary becomes non-absorbing and I did not succeed in proving the uniqueness of the limiting diffusion process. This was finally proved by a beautiful paper of T. Shiga (Math. Soc. Japan, 39 (1987), 17–25).

About population genetics I only studied M. Kimura's introductory book in Japanese (Shûdan-iden-gaku Gairon, Baifûkan, 1960). I did not know much while I was working on genetic models. I wanted more systematic knowledge before deciding my future direction and studied in 1977–1978 the book by J.F. Crow and M. Kimura, An introduction to population genetics theory, Harper and Row, 1970. But it was not exciting to me and I was far more interested in infinitely divisible distributions and Lévy processes. I did not work on genetic models any thereafter.

6. Distributions of class L [20].

When I was back to Tokyo near the end of June, 1976, after the visit to University of Minnesota from fall 1975, I heard from Makoto Yamazato about his proof of unimodality of all distributions of class L on the line. It had a long-lasting impact on me. His key lemma was a great idea and, above all, the fact was overwhelming that he continued to attack the problem that had been unsolvable by many people and really succeeded in the proof, as we checked. The book by B.V. Gnedenko and

[19]{More precisely, the operator for the limiting diffusion process on a $(d-1)$-dimensional simplex in the d-dimensional space.}

[20]{Now distributions of class L are usually called *selfdecomposable* distributions.}

A.N. Kolmogorov contained the "proof" of this unimodality given by A.I. Lapin but, in translating it into English (Limit distributions for sums of independent random variables, Addison-Wesley, 1954), K.L. Chung[21] found that a lemma used by Lapin was erroneous and thus the unimodality is an open problem. Then more than five papers had been written but none of them had proved or disproved it. Stable distributions belong to the class L, but they also had no proof of unimodality in the non-symmetric two-sided case. Unimodality of one-sided distributions of class L was proved by S.J. Wolf[22]. But we did not know it. I remember that Yamazato also proved this before the fall of 1975 and said that the two-sided case was hard.

As Yamazato talked today, he became interested in the class L, since distributions appearing in limit theorems for branching processes were of class L. In my case, after the work by him, I made the study of more detailed properties of distributions of class L on the line, the study of some related classes in general dimensions, and so on, sometimes jointly with him and sometimes independently. With great effort I worked on absolute continuity of multi-dimensional distributions of class L and on characterization of strictly operator-stable distributions.

It is easy to see that distributions of class L on the line are absolutely continuous except the case degenerate to one-point mass. But it had been unknown whether non-degenerate distributions of class L in higher dimensions were absolutely continuous, since their Lévy measures could be singular. I got an idea of some decomposition and proved it in [11.2].

Any stable distribution on the d-dimensional space is a translate of a strictly strictly stable distribution, when and only when its index α is not equal to 1. A stable distribution of index 1 is strictly stable if and only if the spherical component of its Lévy measure has barycenter at the origin. This fact is well known. I was interested what is the corresponding condition in the case of operator-stable distributions and gave a solution of this problem in [13.2]. But it is not known what intuitive meaning this condition has in operator-stable processes and in limit theorems of partial sums of independent identically distributed random variables in d dimensions (originally an operator-stable distribution was introduced by M. Sharpe[23] as the limit distribution of a matrix normalization of partial sums of independent identically distributed random variables in d dimensions).

[21] C. R. Acad. Sci. Paris, 236 (1953), 583–584.
[22] Ann. Math. Stat., 42 (1971), 912–918.
[23] Trans. A. M. S., 136 (1969), 51–65.

In [17.1] and [17.2] I gave a characterization of any distribution of class L in d dimensions as the distribution at a fixed time of a selfsimilar additive process (here an additive process means a process continuous in probability having independent increments and starting at 0). I think this expresses the essence of the class L, although the proof is simple. It follows from this that two processes correspond to each distribution of class L — one is a selfsimilar additive process and the other is a Lévy process; both of them have the given distribution at time 1. I named the former as a process of class L [24] and the latter as a selfdecomposable process. The study of the former processes has hardly begun.

The characterization [12.2] of distributions of class L and, more generally, operator selfdecomposable distributions as the limit distributions of Ornstein–Uhlenbeck type processes was obtained in the middle of 1981 as I was, jointly with Yamazato, seeking the meaning of the equation that appeared in the analysis of class L; we also had S.J. Wolfe's preprint[25] kindly sent to us. Almost at the same time the work of Z.J. Jurek and W. Vervaat and the subsequent work of Wolfe obtained similar results. However, only our group is interested in Ornstein–Uhlenbeck type processes themselves such as criterion of their recurrence and transience. The criterion was given by T. Shiga[26] in one dimension and, in higher dimensions, [18.2] was written by Yamazato, Toshiro Watanabe, and me and [18.3] by those three joined by Kôji Yamamuro, but there still remains an unsolved part.

7. Lévy processes[27].

If a Lévy process X_t on the line has a distribution of class L at some t, then it does so at all t, hence unimodal always. But a general Lévy process is far more complicated; its distribution may have time evolution, such as change from unimodal to non-unimodal or from non-unimodal to unimodal, or change repeatedly. Simple examples were remarked by Wolfe[28]. I thought it was important to examine this phenomenon more deeply and talked it in *[20.5]. I stress that the distribution of a Lévy process can have qualitative time evolution, non-linear time evolution although the increments of the process are homogeneous in time.

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[24]{This naming "process of class L" is not used these days.}

[25]{This appeared later in Stoch. Proc. Appl., 12 (1982), 301–312.}

[26]Prob. Th. Rel. Fields, 85 (1990), 425–447.

[27]{In the original Japanese text, Lévy process is called *kahou katei*. The direct translation of this word is additive process, but this was used in the meaning of the process now called Lévy process.}

[28]Zeit. Wahrsch., 45 (1978), 329–335.

In [20.3] I constructed, for an arbitrary positive integer n, a Lévy process on the line which is unimodal at time 1 and n-modal at time 2. In the proof I made use of F.W. Steutel's result[29] that any distribution on the half line with log-convex density is infinitely divisible.

Concerning absolute continuity, results of H. Rubin and H.G. Tucker[30] imply the existence of a Lévy process on the line having a critical time t_0 such that its distribution is continuous singular for $t < t_0$ and absolutely continuous for $t > t_0$. I added a number of such examples in [20.2].

Another remarkable work[31] of Yamazato is the study of strong unimodality, which is meaningful also from the view point of time evolution. It is sometimes powerful in analyzing concrete Lévy processes on the line.

Toshiro Watanabe started, about 1988, intensive research on distributional properties of Lévy processes. I was happy to follow his works and to gain a new impetus. He treated various examples for unimodality and movement of mode with involved and ingenious analysis and, up to now, wrote seven papers beginning with the one in Jap. J. Math., 15 (1989), 191–203. He found a new method which is an effective use of the transformation of distributions on $[0, \infty)$ to discrete distributions (mixtures of Poisson distributions) introduced by G. Forst[32] and others. I named this transformation as Poisson transform in *[20.5] and gave an exposition in *[21.6].

On Lévy processes I gathered results systematically and wrote a book *[21.5].

There are many interesting objects other than Lévy and Ornstein–Uhlenbeck type processes. I would like to work on them in the future. I would appreciate your support.

Part II

Here is a list of my mathematical papers up to now. Some items other than refereed papers are added with asterisks * .

1. Existence of Markov processes.

[1.1] Integration of the generalized Kolmogorov-Feller backward equations. J. Fac. Sci. Univ. Tokyo, Sect. I, Vol. 9, 13–27. (1961)

[29] Math. Centre Tracts, No. 33, Amsterdam, 1970.
[30] Trans. A. M. S., 118 (1965), 316–330.
[31] Ann. Prob., 10 (1982), 589–601.
[32] Zeit. Wahrsch., 49 (1979), 349–352.

[1.2] Lévy measures for a class of Markov processes in one dimension. Trans. Amer. Math. Soc., Vol. 148, 211–231. (1970)

2. Boundary behavior problem of Markov processes.

[2.1] (With H. Tanaka) Local times on the boundary for multi-dimensional reflecting diffusion. Proc. Japan Acad., Vol. 38, 699–702. (1962)

[2.2] Time change and killing for multi-dimensional reflecting diffusion. Proc. Japan Acad., Vol. 39, 69–73. (1963)

[2.3] (With T. Ueno) Multi-dimensional diffusion and the Markov process on the boundary. J. Math. Kyoto Univ., Vol. 4, 529–605. (1965)

[2.4] A decomposition of Markov processes. J. Math. Soc. Japan, Vol. 17, 219–243. (1965)

*[2.5] (With M Motoo) Estimate of fractional powers of elliptic differential operators and its application to the problem of sums of infinitesimal generators (in Japanese). Abstracts of contributed talks, Math. Soc. Japan, Section on probability and statistics (May 1967), p. 5.

*[2.6] Existence of diffusion processes satisfying Wentzell's boundary condition containing second-order terms (in Japanese). Abstracts of contributed talks, Math. Soc. Japan, Section on probability and statistics (May 1967), p. 6.

*[2.7] Diffusion processes with general boundary conditions and Ueno's processes. Seminar handout at University of Minnesota, 10 pages. (1967)

3. General theory of Markov processes. (Additive functionals, time reversal.)

[3.1] (With M. Nagasawa) Remarks to "The adjoint processes of diffusions with reflecting barrier". Kôdai. Math. Sem. Rep., Vol. 14, 119–122. (1962)

[3.2] (With M. Nagasawa) Some theorems on time change and killing of Markov processes. Kôdai. Math. Sem. Rep., Vol. 15, 195–219. (1963)

[3.3] (With N. Ikeda and M. Nagasawa) A time reversion of Markov processes with killing. Kôdai. Math. Sem. Rep., Vol. 16, 88–97. (1964)

*[3.4] Semigroups and Markov processes. Lecture Notes, Dept. Math., Univ. Minnesota. (1968)

4. Banach lattices. Characterization of infinitesimal generators. Sums of infinitesimal generators.

[4.1] On the generators of nonnegative contraction semigroups in Banach lattices. J. Math. Soc. Japan, Vol. 20, 423–436. (1968)

[4.2] (With K. Gustafson) Some perturbation theorems for nonnegative contraction semigroups. J. Math. Soc. Japan, Vol. 21, 200–204. (1969)

[4.3] Positive pseudo-resolvents in Banach lattices. J. Fac. Sci. Univ. Tokyo, Sec. I, Vol. 17, 305–313. (1970)

[4.4] On dispersive operators in Banach lattices. Pacific J. Math., Vol. 33, 429–443. (1970)

[4.5] A note on nonlinear dispersive operators. J. Fac. Sci. Univ. Tokyo, Sec. IA, Vol. 18, 465–473. (1972)

5. Potential operators.

[5.1] Potential operators for Markov processes. Proc. Sixth Berkeley Symp. Math. Stat. and Prob., Vol. 3, Univ. Calif. Press, 193–211. (1972)

[5.2] A note on infinitesimal generators and potential operators of contraction semigroups. Proc. Japan Acad., Vol. 48, 450–453. (1972)

[5.3] Cores of potential operators for processes with stationary independent increments. Nagoya Math. J., Vol. 48, 129–145. (1972)

*[5.4] Potential operators for Markov processes (in Japanese). Theory of Markov processes, RIMS Kôkyûroku, No. 112, 55–79. (1971)

6. Tails of infinitely divisible distributions.

[6.1] A note on infinitely divisible distributions and their Lévy measures. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, Vol. 12, 101–109. (1973)

7. Population-genetic models. (Estimate of eigenvalues. Large deviations.)

[7.1] Asymptotic properties of eigenvalues of a class of Markov chains induced by direct product branching processes. J. Math. Soc. Japan, Vol. 28, 192–211. (1976)

*[7.2] Asymptotic properties of Markov chains arising in population genetics, and limit theorems for large deviations (in Japanese). Studies of Markov processes, Reports of Symposium in January 1975, Seminar on Probability, Vol. 41, 93–102. (1975)

8. Population-genetic models. (Diffusion approximation. Natural selection.)

[8.1] Diffusion processes and a class of Markov chains related to population genetics. Osaka J. Math., Vol. 13, 631–659. (1976)

[8.2] A class of Markov chains related to selection in population genetics. J. Math. Soc. Japan, Vol. 28, 621–637. (1976)

[8.3] Convergence to diffusion processes for a class of Markov chains related to population genetics. Proc. Third Japan–USSR Symp. on Prob. Th., Lect. Notes in Math., Springer, No. 550, 550–561. (1976)

[8.4] A note on convergence of probability measures on C and D. Ann. Sci. Kanazawa Univ., Vol. 14, 1–5. (1977)

[8.5] Convergence of a class of Markov chains to multi-dimensional degenerate diffusion processes. Proc. Internat. Symp. on Stoch. Diff. Eq., Kinokuniya, 367–383. (1978)

[8.6] Convergence to a diffusion of a multi-allelic model in population genetics. Adv. Appl. Prob., Vol. 10, 538–562. (1978)

[8.7] Diffusion operators in population genetics and convergence of Markov chains. Measure Theory, Applications to Stoch. Analysis, Lect. Notes in Math., Springer, No. 695, 127–137. (1978)

[8.8] Limit diffusions of some stepping-stone models. J. Appl. Prob., Vol. 20, 460–471. (1983)

9. Fine properties of one-dimensional distributions of class L.

[9.1] (With M. Yamazato) On distribution functions of class L. Zeitsch. Wahrscheinlich. verw. Gebiete, Bd. 43, 273–308. (1978)

[9.2] (With M. Yamazato) On higher derivatives of distribution functions of class L. J. Math. Kyoto Univ., Vol. 21, 575–591. (1981)

*[9.3] On distributions of class L (in Japanese). Studies of Markov processes, Reports of Kanazawa Symposium in December 1976, Seminar on Prob., Vol. 44, 147–162. (1977)

10. Subclasses of the class L.

[10.1] Urbanik's class L_m of probability measures. Ann. Sci. Kanazawa Univ., Vol. 15, 1–10. (1978)

[10.2] Class L of multivariate distributions and its subclasses. J. Multivar. Anal., Vol. 10, 207–232. (1980)

11. Absolute continuity of multi-dimensional distributions of class L.

[11.1] On densities of multivariate distributions of class L. Ann. Sci. Kanazawa Univ., Vol. 16, 1–9. (1979)

[11.2] Absolute continuity of multivariate distributions of class L. J. Multivar. Anal., Vol. 12, 89–94. (1982)

12. Relations of distributions of class L with Ornstein–Uhlenbeck type processes.

[12.1] (With M. Yamazato) Stationary processes of Ornstein–Uhlenbeck type. Probability Theory and Math. Statistics, Fourth USSR-Japan Symp., Lect. Notes in Math., Springer, No. 1021, 541–551. (1983)

[12.2] (With M. Yamazato) Operator-self-decomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type. Stoch. Proc. Appl., Vol. 17, 73–100. (1984)

13. Operator-stable distributions.

[13.1] (With M. Yamazato) Completely operator-self-decomposable distributions and operator-stable distributions. Nagoya Math. J., Vol. 97, 71–94. (1985)

[13.2] Strictly operator-stable distributions. J. Multivar. Anal., Vol. 22, 278–295. (1987)

*[13.3] Lectures on multivariate infinitely divisible distributions and operator-stable processes. Tech. Rep. Ser. Lab. Res. Stat. Prob., Carleton Univ. and Univ. Ottawa, No. 54. (1985)

14. Behavior of modes of Lévy processes.

[14.1] Bounds of modes and unimodal processes with independent increments. Nagoya Math. J., Vol. 104, 29–42. (1986)

[14.2] Behavior of modes of a class of processes with independent increments. J. Math. Soc. Japan, Vol. 38, 679–695. (1986)

15. General theory of bounds of modes.

[15.1] Modes and moments of unimodal distributions. Ann. Inst. Stat. Math., Vol. 39, 407–415. (1987)

16. Unimodality of functionals of birth-and-death processes.

[16.1] Unimodality and bounds of modes for distributions of generalized sojourn times. Stochastic Methods in Biology, Lect. Notes in Biomath., Springer, No. 70, 210–221. (1987)

[16.2] Some classes generated by exponential distributions. Probability Theory and Math. Statistics, Fifth Japan–USSR Symp., Lect. Notes in Math., Springer, No. 1299, 454–463. (1988)

[16.3] On zeros of a system of polynomials and application to sojourn time distributions of birth-and-death processes. Trans. Amer. Math. Soc., Vol. 309, 375–390. (1988)

17. Distributions of class L and selfsimilar additive processes.

[17.1] Distributions of class L and self-similar processes with independent increments. White Noise Analysis. Mathematics and Applications, World Scientific, 360–373. (1990)

[17.2] Self-similar processes with independent increments. Prob. Th. Rel. Fields, Vol. 89, 285–300. (1991)

18. Recurrence criteria for Ornstein–Uhlenbeck type processes.

[18.1] (With M. Yamazato) Remarks on recurrence criteria for processes of Ornstein–Uhlenbeck type. Functional Analysis and Related Topics, 1991, Lect. Notes in Math., Springer, No. 1540, 329–340. (1993)

[18.2] (With Toshiro Watanabe and M. Yamazato) Recurrence conditions for multidimensional processes of Ornstein–Uhlenbeck type. J. Math. Soc. Japan, Vol. 46, 245–265. (1994)

[18.3] (With Toshiro Watanabe, K. Yamamuro, and M. Yamazato) Multidimensional process of Ornstein–Uhlenbeck type with nondiagonalizable matrix in linear drift terms. Nagoya Math. J., {Vol. 141, 45–78. (1996)}

19. General theory of convolutions of unimodal distributions.

[19.1] Convolution of unimodal distributions can produce any number of modes. Ann. Prob., Vol. 21, 1543–1549. (1993)

20. Time evolution of distributions of Lévy processes.

[20.1] On unimodality and mode behavior of Lévy processes. Probability Theory and Mathematical Statistics, Proc. Sixth USSR–Japan Symp., World Scientific, 292–305. (1992)

[20.2] Time evolution of distributions of Lévy processes from continuous singular to absolutely continuous. Research Bulletin, College of General Education, Nagoya Univ., Ser. B, No. 38, 1–11. (1994)

[20.3] Multimodal convolutions of unimodal infinitely divisible distributions. Teor. Veroyatn. Primenen., {Tom 39, 403–415 (Theory Probab. Appl., Vol. 39, 336–347). (1994)}

[20.4] Time evolution in distributions of Lévy processes. Southeast Asian Bulletin of Mathematics, {Vol. 19, No. 2, 17–26. (1995)}

*[20.5] Problems on unimodality of distributions of Lévy processes (in Japanese). Abstracts of talks, Math. Soc. Japan, Section on probability and statistics (October 1992), 57–70.

21. Others.

[21.1] Subordination depending on a parameter. Probability Theory and Mathematical Statistics, Proc. Fifth Vilnius Conf., Vol. 2, VSP/Mokslas, 372–382. (1990)

[21.2] (With M. Fukushima and S. Taniguchi) On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures. Osaka J. Math., Vol. 28, 517–535. (1991)

*[21.3] Infinitely divisible distributions (in Japanese). Seminar on Probability, Vol. 52. (1981)

*[21.4] Properties of distribution of passage time (in Japanese). Stochastic Models in Population genetics, Reports of Symposium, 84–96. (1985)

*[21.5] Lévy Processes (in Japanese). Kinokuniya. (1990)

*[21.6] Poisson transform (from continuous distributions to discrete distributions) (in Japanese). Topics in distribution theory, Abstracts, 13–24. (1992)