ON A THEOREM ON LÉVY PROCESSES

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Let $\{X_t \colon t \geqslant 0\}$ be a Lévy process on $\mathbb{R}$ with generating triplet (A,ν,γ), that is,

$$Ee^{izX_t} = \exp\left[t\left(-\frac{1}{2}Az^2 + \int_{\mathbb{R}}(e^{izx} - 1 - izx1_{[-1,1]}(x))\,\nu(dx) + i\gamma z\right)\right],$$

where A is the Gaussian variance and ν is the Lévy measure of $\{X_t\}$. In Sato (2001), p. 22, the author writes:

> Let us consider the behavior of $t^{-1}X_t$ as $t \to \infty$. The study of random walks by Kesten (1970) and Erickson (1973) has the following analogue in Lévy processes on $\mathbb{R}$. Let m^+ and m^- be as defined below Corollary 3.1 in Section 3. [That is, $m^+ = \int_{(0,\infty)} x\mu(dx)$ and $m^- = \int_{(-\infty,0)} x\mu(dx)$, where μ is the distribution of X_1.]
>
> **Theorem 5.8.** *Suppose that $m^+ = \infty$ and $m^- = -\infty$. Then there are only three possibilities:*
> (1) $\lim_{t\to\infty} t^{-1}X_t = \infty$ *a. s.;*
> (2) $\lim_{t\to\infty} t^{-1}X_t = -\infty$ *a. s.;*
> (3) $\limsup_{t\to\infty} t^{-1}X_t = \infty$ *and* $\liminf_{t\to\infty} t^{-1}X_t = -\infty$ *a. s.*
> *Let*
>
> $$K^+ = \int_{(2,\infty)} x\left(\int_{-x}^{-1} \nu((-\infty,y))dy\right)^{-1}\nu(dx),$$
> $$K^- = \int_{(-\infty,-2)} |x|\left(\int_{1}^{|x|} \nu((y,\infty))dy\right)^{-1}\nu(dx),$$
>
> *where ν is the Lévy measure of $\{X_t\}$. Then $K^+ + K^- = \infty$ and the following equivalences are true:* (1) *holds if and only if $K^+ = \infty$ and $K^- < \infty$;* (2) *holds if and only if $K^+ < \infty$ and $K^- = \infty$;* (3) *holds if and only if $K^+ = \infty$ and $K^- = \infty$.*

Under the condition that $m^+ = \infty$ and $m^- = -\infty$, the properties (1), (2), and (3) above are respectively equivalent to the drifting to ∞, the drifting to $-\infty$, and the oscillating. Since m^+ and m^- are expressed by the generating triplet (A,ν,γ), Theorem 5.8 means that we have now the classification into the drifting to ∞, to $-\infty$, and the oscillating in terms of (A, ν, γ).

(end of quote from Sato (2001))

Theorem 5.8 and its proof are suggested in Erickson (1973), p. 373. Here we will give a proof in detail. Then we will give some examples.

We use the following theorems of Kesten (1970) and Erickson (1973) (see also Remark 36.10 of Sato (1999)). Let $\{S_n\colon n=0,1,\dots\}$ be a random walk on $\mathbb{R}$, that is, $S_0=0$ and $S_n = Z_1+\cdots+Z_n$, where $\{Z_j\colon j=1,2,\dots\}$ is independent and identically distributed.

Theorem A. *If $E[S_1\vee 0]=\infty$ and $E[S_1\wedge 0]=-\infty$, then one of the following three cases necessarily occurs:*

Case 1*:* $\lim_{n\to\infty} n^{-1}S_n = \infty$ *a. s.;*

Case 2*:* $\lim_{n\to\infty} n^{-1}S_n = -\infty$ *a. s.;*

Case 3*:* $\limsup_{n\to\infty} n^{-1}S_n = \infty$ *and* $\liminf_{n\to\infty} n^{-1}S_n = -\infty$ *a. s.*

Theorem B. *Assume that $E[S_1\vee 0]=\infty$ and $E[S_1\wedge 0]=-\infty$. Let ρ be the distribution of S_1 and define*

$$J^+ = \int_{(0,\infty)} x\left(\int_{-x}^{0}\rho(-\infty,y)dy\right)^{-1}\rho(dx),$$

$$J^- = \int_{(-\infty,0)} |x|\left(\int_{0}^{|x|}\rho(y,\infty)dy\right)^{-1}\rho(dx).$$

Then,

$$\begin{aligned} J^+=\infty \quad &\textit{and} \quad J^-<\infty \quad \Longleftrightarrow \quad \textit{Case } 1,\\ J^+<\infty \quad &\textit{and} \quad J^-=\infty \quad \Longleftrightarrow \quad \textit{Case } 2,\\ J^+=\infty \quad &\textit{and} \quad J^-=\infty \quad \Longleftrightarrow \quad \textit{Case } 3. \end{aligned}$$

Now Theorem 5.8 quoted above follows from these two theorems.

Proof of Theorem 5.8. The assumptions $m^+=\infty$ and $m^-=-\infty$ are respectively equivalent to $\int_{(1,\infty)} x\nu(dx)=\infty$ and $\int_{(-\infty,0)} x\nu(dx) = -\infty$ (see Theorem 25.3 and Proposition 25.4 of Sato (1999)). Let $\{X'_t\colon t\geqslant 0\}$ and $\{X''_t\colon t\geqslant 0\}$ be independent Lévy processes with generating triplets $(0,[\nu]_{\{|x|>1\}},0)$ and $(A,[\nu]_{\{|x|\leqslant 1\}},\gamma)$, respectively. Here $[\nu]_D$ denotes the restriction of ν to a set D. Then $\{X'_t+X''_t\}$ is identical in law with $\{X_t\}$, so that it is enough to prove the theorem for $\{X'_t + X''_t\}$ in place of $\{X_t\}$. The process $\{X''_t\}$ satisfies $E|X''_1| < \infty$ and hence $\lim_{t\to\infty} t^{-1}X''_t = EX''_1$ a. s. by the strong law of large numbers (see Theorem 36.5 of Sato (1999)). Thus it is enough to prove the theorem for $\{X'_t\}$. Notice that the quantities K^+ and K^- for $\{X'_t\}$ are exactly those for $\{X_t\}$. Let $c = \nu(\{|x| > 1\})$. Since the process $\{X'_t\}$ is a compound Poisson process, we may assume $X'_t = S_{N_t}$, where $\{S_n\}$ is a random walk with the distribution of S_1 being $\rho = c^{-1}[\nu]_{\{|x|>1\}}$, $\{N_t\}$ is a Poisson process with parameter c, and $\{S_n\}$ and $\{N_t\}$ are independent. We have $\lim_{t\to\infty} t^{-1}N_t = c$ a. s. Thus

$$\limsup_{t\to\infty} \frac{X'_t}{t} = \limsup_{t\to\infty} \frac{N_t}{t}\frac{S_{N_t}}{N_t} = c \limsup_{n\to\infty} \frac{S_n}{n}$$

and similarly

$$\liminf_{t\to\infty} \frac{X'_t}{t} = c \liminf_{n\to\infty} \frac{S_n}{n}.$$

Notice that $E[S_1 \vee 0] = \infty$ and $E[S_1 \wedge 0] = -\infty$. Thus, by Theorem A, one of Cases 1, 2, and 3 must occur concerning the behavior of $\{n^{-1}S_n\}$. These correspond to (1), (2), and (3), respectively, concerning the behavior of $\{t^{-1}X_t\}$. We claim that

$$(*) \qquad K^+ = \infty \iff J^+ = \infty,$$

$$(**) \qquad K^- = \infty \iff J^- = \infty.$$

These equivalences will establish the latter half of Theorem 5.8. Let, for $x > 1$,

$$M^+(x) = \int_1^x \nu((y,\infty))dy, \quad M^-(x) = \int_1^x \nu((-\infty,-y))dy = \int_{-x}^{-1} \nu((-\infty,y))dy.$$

Since $[\nu]_{\{|x|>1\}} = c\rho$, we have, for $x > 1$,

$$M^+(x) = c\int_1^x \rho((y,\infty))dy = c\int_0^x \rho((y,\infty))dy - c\rho((1,\infty)),$$
$$M^-(x) = c\int_{-x}^{-1} \rho((-\infty,y))dy = c\int_{-x}^{0} \rho((-\infty,y))dy - c\rho((-\infty,-1)).$$

Note that

$$K^+ = \int_{(2,\infty)} \frac{x}{M^-(x)}\nu(dx), \quad K^- = \int_{(-\infty,-2)} \frac{|x|}{M^+(|x|)}\nu(dx).$$

It follows from $\int_{(1,\infty)} x\nu(dx) = \infty$ and $\int_{(-\infty,-1)} x\nu(dx) = -\infty$ that $M^+(\infty) = \infty$ and $M^-(\infty) = \infty$. Hence,

$$\begin{aligned} K^+ = \infty \quad &\iff \quad \int_{(2,\infty)} \frac{x\nu(dx)}{M^-(x) + c\rho((-\infty,-1))} = \infty \\ &\iff \quad \int_{(2,\infty)} \frac{x\rho(dx)}{\int_{-x}^0 \rho((-\infty,y))dy} = \infty \\ &\iff \quad J^+ = \infty \end{aligned}$$

and, similarly,

$$\begin{aligned} K^- = \infty \quad &\iff \quad \int_{(-\infty,-2)} \frac{|x|\nu(dx)}{M^+(|x|) + c\rho((1,\infty))} = \infty \\ &\iff \quad \int_{(-\infty,-2)} \frac{|x|\rho(dx)}{\int_0^{|x|} \rho((y,\infty))dy} = \infty \\ &\iff \quad J^- = \infty. \end{aligned}$$

Thus we get (*) and (**). The assertion $K^+ + K^- = \infty$ is a consequence of the other assertions, but its direct proof can be obtained as in p. 376 of Erickson (1973). This finishes a proof of Theorem 5.8.

As is written below Theorem 5.8, under the assumption that $m^+ = \infty$ and $m^- = -\infty$,

$$\begin{aligned} (1) \quad &\iff \quad \lim_{t\to\infty} X_t = \infty \text{ a.s. (drifting to } \infty); \\ (2) \quad &\iff \quad \lim_{t\to\infty} X_t = -\infty \text{ a.s. (drifting to } -\infty); \\ (3) \quad &\iff \quad \limsup_{t\to\infty} X_t = \infty \text{ and } \liminf_{t\to\infty} X_t = -\infty \text{ a.s. (oscillating).} \end{aligned}$$

Indeed, the implication from the left to the right is obvious. Since the three conditions on the right are mutually exclusive and since (1), (2), and (3) are exhaustive by virtue of Theorem 5.8, the implication from the right to the left follows.

Examples. Let us consider some examples. In each of them the left tail of the Lévy measure is strictly fatter than the right tail. Some of them satisfy (3) while the others satisfy (2). In the following, c_1 and c_2 are positive constants.

(i) Let $0 < \beta < \alpha \leqslant 1$. Let

$$\nu(dx) = \begin{cases} c_1 x^{-1-\alpha} dx & \text{for large } x \\ c_2 |x|^{-1-\beta} dx & \text{for large } |x| \text{ with } x < 0. \end{cases}$$

Then $K^+ < \infty$ and $K^- = \infty$. Hence (2) holds, that is, $\lim_{t\to\infty} t^{-1}X_t = -\infty$. If, moreover, $\alpha < 1$, then drifting to $-\infty$ of this example is shown in Example 48.5 of Sato (1999) by another method.

(ii) Let $0 < \alpha < 1$ and $\beta > 0$. Let

$$\nu(dx) = \begin{cases} c_1 x^{-1-\alpha} dx & \text{for large } x \\ c_2 |x|^{-1-\alpha} (\log |x|)^\beta \, dx & \text{for large } |x| \text{ with } x < 0. \end{cases}$$

If $\beta > 1$, then $K^+ < \infty$ and $K^- = \infty$, and thus (2) holds. If $\beta \leqslant 1$, then $K^+ = \infty$ and $K^- = \infty$, and hence (3) holds and oscillating.

(iii) Let

$$\nu(dx) = \begin{cases} c_1 x^{-2} dx & \text{for large } x \\ c_2 |x|^{-2} (\log |x|)^\beta \, dx & \text{for large } |x| \text{ with } x < 0 \end{cases}$$

with $\beta > 0$. Then $K^+ < \infty$ and $K^- = \infty$. Hence (2) holds.

(iv) (Erickson (1973) p. 374 for $\beta = 1$) Let $\beta > 0$ and let

$$\nu(dx) = \begin{cases} c_1 x^{-2} dx & \text{for large } x \\ c_2 |x|^{-2} (\log \log |x|)^\beta \, dx & \text{for large } |x| \text{ with } x < 0. \end{cases}$$

If $\beta > 1$, then $K^+ < \infty$ and $K^- = \infty$, and thus (2) holds. If $\beta \leqslant 1$, then $K^+ = \infty$ and $K^- = \infty$, and thus (3) holds.

Remark 1. For the examples that satisfy (3), a question whether they are recurrent or transient arises. The author believes the case $0 < \beta \leqslant 1$ of the example (ii) is transient. But he has no conjecture in the case $0 < \beta \leqslant 1$ of (iv).

Remark 2. There are regrettable errors in the last two lines in p. 256 of Sato (1999). The definitions of K^+ and K^- there should be changed to those in Theorem 5.8 of Sato (2001) quoted above.

References

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