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STOCHASTIC INTEGRALS WITH RESPECT TO LÉVY PROCESSES AND INFINITELY DIVISIBLE DISTRIBUTIONS

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ABSTRACT. Results on two topics on stochastic integrals with respect to Lévy processes and infinitely divisible distributions are surveyed. The first topic is on the distributions of Poissonian exponential integrals with four parameters, which are stationary distributions of some generalized Ornstein–Uhlenbeck processes. The classification of the distributions is made according to infinitely divisible or not infinitely divisible and according to quasi-infinitely divisible or not quasi-infinitely divisible. Their classification according to absolutely continuous or continuous-singular is pursued. The second topic is on the transformation of infinitely divisible distributions via improper stochastic integrals of non-random functions with respect to Lévy processes. Related subclasses and sequences of subclasses of the class of infinitely divisible distributions are studied. The results on the first topic are by Lindner and Sato (2009, 2011) and those on the second are by several papers of Barndorff-Nielsen, Jurek, Maejima, and Sato.

1. BASIC FACTS

This article treats two topics on stochastic integrals with respect to Lévy processes and infinitely divisible distributions. The first topic is on the distributions of stochastic integrals called Poissonian exponential integrals; we study in a four-parameter case whether they are infinitely divisible or not and whether they are absolutely continuous or continuous-singular and clarify the dependence of those properties on the parameters. The second topic is on the transformation of infinitely divisible distributions via stochastic integrals with respect to Lévy processes; we give representations of various known subclasses of the class of infinitely divisible distributions and pursue new subclasses.

Let us begin with some basic definitions. A probability space $(\Omega, \mathcal{F}, P)$ is the triplet of a set Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω, and a measure P with total mass 1. A random variable X taking values in the Euclidean space $\mathbb{R}^d$ is an $\mathcal{F}$-measurable mapping from Ω into $\mathbb{R}^d$. A stochastic process $\{X_t : t \geq 0\}$ is a family of random variables with time parameter t. An $\mathbb{R}^d$-valued *Lévy process* $\{X_t : t \geq 0\}$ is a stochastic process having independent increments, being time-homogeneous and continuous in probability, and starting at 0, that is, a stochastic process satisfying

(1) for any $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,
(2) the distribution of $X_{s+t} - X_s$ does not depend on s,
(3) $X_0 = 0$ almost surely (a. s.),
(4) for any $t \geq 0$ and $\varepsilon > 0$, $\lim_{s \to t} P[|X_s - X_t| > \varepsilon] = 0$.

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A stochastic process $\{X_t\colon t \geq 0\}$ is called an *additive process* if it satisfies (1), (3), and (4) (namely, time-homogeneity (2) is not assumed).

If $\{X_t\colon t \geq 0\}$ is a Lévy or additive process, then we can modify, for each t, the random variable X_t on a subset of Ω with probability 0 so that the path $X_t(\omega)$ is right-continuous with left limits as a function of t. Hence, from now on, we assume that this property is satisfied when we say that $\{X_t\colon t \geq 0\}$ is a Lévy or additive process.

The class of Lévy processes is the most important class in the theory of stochastic processes. Brownian motion and the Poisson process belong to this class. A deep study of general Lévy and additive processes was made by P. Lévy [22] in the 1930s. But it was around the middle of the 1990s that the name Lévy process was established. The usage of the name additive process is not yet fixed. On the one hand some people in Japan call a Lévy process in our sense an additive process; Sato's book [42] in Japanese followed this naming. On the other hand, there are some who call our additive process a Lévy process and they call our Lévy process a homogeneous Lévy process.

The inner product of $x, y \in \mathbb{R}^d$ and the norm of $x \in \mathbb{R}^d$ are denoted by $\langle x, y\rangle$ and $|x|$. For real numbers a and b we use $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. The characteristic function of a distribution (that is, probability measure) μ on $\mathbb{R}^d$ is the Fourier transform $\widehat{\mu}(z) = \int e^{i\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$. The convolution $\mu * \rho$ of distributions μ and ρ on $\mathbb{R}^d$ is the distribution whose characteristic function equals $\widehat{\mu}(z)\widehat{\rho}(z)$. If μ and ρ are distributions (laws) of independent random variables X and Y, respectively, then $\mu * \rho$ is the distribution of $X + Y$. A distribution μ is called *infinitely divisible* if, for each positive integer n, there is a distribution μ_n such that $\widehat{\mu}(z) = \widehat{\mu}_n(z)^n$.

Here are two basic results.

Theorem 1.1. *If $\{X_t\colon t \geq 0\}$ is an $\mathbb{R}^d$-valued Lévy process, then, for any t, the distribution of X_t is infinitely divisible. Conversely, for any infinitely divisible distribution μ, there uniquely (in the sense of law) exists a Lévy process $\{X_t\colon t \geq 0\}$ such that X_1 (namely X_t with $t = 1$) has distribution μ.*

Moreover, if $\{X_t\colon t \geq 0\}$ is an $\mathbb{R}^d$-valued additive process, then, for any t, the distribution of X_t is infinitely divisible. If μ is infinitely divisible, then the characteristic function $\widehat{\mu}(z)$ does not vanish for any $z \in \mathbb{R}^d$. If μ is a distribution with $\widehat{\mu}(z)$ having no zero, then there is a unique continuous function $C_\mu(z)$ satisfying $\widehat{\mu}(z) = \exp(C_\mu(z))$ and $C_\mu(0) = 0$; this function $C_\mu(z)$ is called the cumulant function of μ.

Theorem 1.2 (Lévy–Khintchine representation)**.** *If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then*

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx) + i\langle \gamma, z\rangle, \tag{1.1}$$

where A is a non-negative definite symmetric $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int(|x|^2 \wedge 1)\nu(dx) < \infty$, and $\gamma \in \mathbb{R}^d$. The triplet (A, ν, γ) is uniquely determined by μ. Conversely, for any triplet (A, ν, γ), there uniquely exists an infinitely divisible distribution μ on $\mathbb{R}^d$ satisfying (1.1).

The measure ν in this theorem is called the *Lévy measure* of μ. Theorem 1.1 shows that there is a one-one correspondence between infinitely divisible distributions μ and Lévy processes $\{X_t\}$ in such a way that X_1 has distribution μ. The triplet (A,ν,γ) of μ is called the *generating triplet* of $\{X_t\}$; sometimes it is written as (A_X,ν_X,γ_X). If the generating triplet is $(I,0,0)$ (where I is the unit matrix), that is, if μ is the standard Gaussian distribution, then $\{X_t\}$ is the $\mathbb{R}^d$-valued Brownian motion and the path is continuous. A distribution concentrated at a point γ (one-point distribution) is denoted by δ_γ, which is infinitely divisible with triplet $(0,0,\gamma)$ corresponding to the deterministic motion $X_t = t\gamma$. If $d=1$, $A=0$, $\nu = a\delta_1$ with $a>0$, and $\widehat{\mu}(z) = \exp[a(e^{iz}-1)]$, that is, if μ is a Poisson distribution with mean a, then $\{X_t\}$ is a Poisson process with parameter a and the path increases only by jumps of height 1.

Let $H>0$. An $\mathbb{R}^d$-valued stochastic process $\{X_t\colon t\geq 0\}$ is called *H-selfsimilar* if, for any $a>0$, $\{X_{at}\colon t\geq 0\} \overset{\mathrm{d}}{=} \{a^H X_t\colon t\geq 0\}$ (the stochastic processes on both sides have identical finite-dimensional joint distributions). A process $\{X_t\colon t\geq 0\}$ is called *broad-sense H-selfsimilar* if, for any $a>0$, there is an $\mathbb{R}^d$-valued function $c_a(t)$ such that $\{X_{at}\colon t\geq 0\} \overset{\mathrm{d}}{=} \{a^H X_t + c_a(t)\colon t\geq 0\}$. A broad-sense H-selfsimilar Lévy process is called a *stable process*. An H-selfsimilar Lévy process is called a *strictly stable process*. If $\{X_t\}$ is a stable process but not $X_t = t\gamma$ or if $\{X_t\}$ is a strictly stable process but not $X_t = 0$, then H is unique and $H\geq 1/2$. Let $\alpha = 1/H$ so that $0<\alpha\leq 2$; α is called the index of the stable or strictly stable process. The infinitely divisible distribution corresponding to a stable or strictly stable process with index α is called a stable or strictly stable distribution with index α. For example, Cauchy distribution is strictly stable with index 1.

A distribution μ on $\mathbb{R}^d$ is called *selfdecomposable* or of class L if, for any $0<b<1$, there is a distribution ρ (dependent on b) such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z). \tag{1.2}$$

If μ is selfdecomposable, then μ is infinitely divisible and ρ is uniquely determined by μ and b and infinitely divisible. For example, exponential and Γ distributions are selfdecomposable on $\mathbb{R}$. The following theorem of Sato [43] shows the importance of selfdecomposable distributions.

Theorem 1.3. *If $\{X_t\colon t\geq 0\}$ is an additive process on $\mathbb{R}^d$ and broad-sense H-selfsimilar for some $H>0$, then, for any t, the distribution of X_t is selfdecomposable. In the converse direction, for any selfdecomposable distribution μ and for any $H>0$, there uniquely (in the sense of law) exists an H-selfsimilar additive process $\{X_t\colon t\geq 0\}$ such that X_1 has distribution μ.*

In this theorem H does not have any importance, because, for any H-selfsimilar additive process $\{X_t\}$ and for any $a>0$, $\{X_{t^a}\}$ is an aH-selfsimilar additive process and the same statement is also true for broad-sense selfsimilar additive processes.

Let $ID(\mathbb{R}^d)$, $S(\mathbb{R}^d)$, and $L(\mathbb{R}^d)$ stand for the classes of infinitely divisible, stable, and selfdecomposable distributions on $\mathbb{R}^d$, respectively. Then

$$ID(\mathbb{R}^d) \supset L(\mathbb{R}^d) \supset S(\mathbb{R}^d). \tag{1.3}$$

Representations of stable and selfdecomposable distributions on $\mathbb{R}^d$ are as follows. Let $\mathcal{B}(\mathbb{R}^d)$ be the class of Borel sets in $\mathbb{R}^d$ and let S be the unit sphere $\{\xi\in\mathbb{R}^d\colon |\xi|=1\}$. The indicator function of a set B is denoted by 1_B. Let $\mu\in ID(\mathbb{R}^d)$ with triplet (A,ν,γ). Then μ is stable with index 2 if and only if $\nu=0$ and A and γ have no restriction (that is, Gaussian distribution); μ is stable with index $0 < \alpha < 2$ if and only if $A = 0$, γ has no restriction, and there is a finite measure λ on S such that

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{1.4}$$

Any stable distribution with index $\alpha \neq 1$ becomes strictly stable if we change γ suitably. This is not true in the case $\alpha = 1$. A stable distribution with index 1 is strictly stable if and only if $\int_S \xi \lambda(d\xi) = 0$. A distribution $\mu \in ID(\mathbb{R}^d)$ is selfdecomposable if and only if there are a measure λ on S and a non-negative function $k_\xi(r)$ measurable in ξ and decreasing and right-continuous in r such that

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-1} k_\xi(r) dr, \quad B \in \mathcal{B}(\mathbb{R}^d), \tag{1.5}$$

with A and γ having no restriction. (The words *increase* and *decrease* are used in the wide sense, allowing flatness.)

Infinitely divisible distributions have a wide variety; their properties are complicated. But, for selfdecomposable distributions on $\mathbb{R}^d$, we have Yamazato's theorem [62] telling their unimodality for $d = 1$ and Sato's result [41] telling their absolute continuity for general d when they are not degenerate to lower dimensions (even if their Lévy measures are not absolutely continuous).

A distribution μ on the half-line $\mathbb{R}_+ = [0, \infty)$ is infinitely divisible if and only if there are a measure ν on $(0, \infty)$ satisfying $\int_{(0,\infty)} (x \wedge 1) \nu(dx) < \infty$ and $\gamma_0 \geq 0$ such that

$$\widehat{\mu}(z) = \exp\left[\int_{(0,\infty)} (e^{izx} - 1)\nu(dx) + i\gamma_0 z\right]. \tag{1.6}$$

Here ν is the Lévy measure of μ; γ_0 is called the *drift* of μ. Such an infinitely divisible distribution corresponds to an increasing Lévy process on $\mathbb{R}$, which is called a *subordinator*.

See Gnedenko and Kolmogorov [15], Loève [28], and Sato [42, 44] for details of the statements above.

Given $b \in (0, 1)$, we say that a distribution μ is *b-decomposable* if (1.2) is satisfied for some ρ. This concept is important in the next section. Such distributions were studied by Loève [27], Wolfe [61], Bunge [8], Watanabe [59], and others. Some b-decomposable distributions μ are not infinitely divisible. However, if μ is b-decomposable having infinitely divisible ρ, then μ is infinitely divisible and is called *semi-selfdecomposable*; semi-stable distributions are examples ([29, 44]). Semi-selfdecomposable distributions can be studied in relation to semi-Lévy processes ([32]). An additive process $\{X_t : t \geq 0\}$ is called a *semi-Lévy process* if its distribution is periodic in the sense that, for some $p > 0$, $X_{t+p} - X_{s+p}$ and $X_t - X_s$ are identically distributed.

2. Properties of Poissonian exponential integrals

Let us begin with some background.

2.1. Generalized Ornstein–Uhlenbeck processes.

Let $\{Y_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}$, M a random variable independent of $\{Y_t\}$, and

$$V_t = e^{-t} \left(M + \int_0^t e^s dY_s \right). \tag{2.1}$$

We call $\{V_t\colon t \geq 0\}$ the *Ornstein–Uhlenbeck process* induced by $\{Y_t\}$. (The integral in the right-hand side of (2.1) is the so-called semimartingale integral. We do not give its definition, but in all cases that appear in Subsections 2.2–2.6 the integral coincides with the pathwise Stieltjes integral.) The process $\{V_t\}$ is the solution of the equation $dV_t = dY_t - V_t dt$ satisfying the initial condition $V_0 = M$. If $\{Y_t\}$ is Brownian motion, then $\{V_t\}$ is called the classical Ornstein–Uhlenbeck process. An Ornstein–Uhlenbeck process is a time-homogeneous Markov process. It is said to have a stationary distribution μ if V_t has distribution μ whenever M has distribution μ. A necessary and sufficient condition for the existence of a stationary distribution μ is that

$$\int_0^{\infty-} e^{-s} dY_s = \lim_{t\to\infty} \int_0^t e^{-s} dY_s \quad \text{is convergent a. s.} \tag{2.2}$$

It is known that this condition is equivalent to the finiteness of $E[\log^+ |Y_1|]$ ($\log^+ a = 0 \vee \log a$) and that this is also equivalent to the finiteness of $\int \log^+ |x| \nu_Y(dx)$. If this condition is satisfied, then the limit in (2.2) has distribution μ and μ is selfdecomposable. Conversely, for any selfdecomposable distribution μ on $\mathbb{R}$, there exists a unique (in the sense of law) Lévy process $\{Y_t\}$ on $\mathbb{R}$ satisfying $E[\log^+ |Y_1|] < \infty$ such that the Ornstein–Uhlenbeck process induced by $\{Y_t\}$ has stationary distribution μ. These facts are also similar in $\mathbb{R}^d$ and provide a characterization of $L(\mathbb{R}^d)$. See [37, 42, 44] for details.

Let us generalize Ornstein–Uhlenbeck processes. Let $\{(X_t, Y_t)\colon t \geq 0\}$ be a Lévy process on $\mathbb{R}^2$. Let $X_{t-}(\omega)$ denote the limit of $X_s(\omega)$ as $s \uparrow t$. A stochastic process $\{V_t\colon t \geq 0\}$ is said to be a *generalized Ornstein–Uhlenbeck process* if

$$V_t = e^{-X_t} \left(M + \int_0^t e^{X_{s-}} dY_s \right) \tag{2.3}$$

with a random variable M independent of $\{(X_t, Y_t)\}$ (Carmona, Petit, and Yor [9, 10]). This is a time-homogeneous real-valued Markov process.

We define a new Lévy process $\{(X_t, L_t)\}$ on $\mathbb{R}^2$ from $\{(X_t, Y_t)\}$ by

$$L_t = Y_t + \sum_{0<s\leq t} (e^{-(X_s - X_{s-})} - 1)(Y_s - Y_{s-}) - t\,\alpha, \tag{2.4}$$

where α is the non-diagonal component of A in the generating triplet of $\{(X_t, Y_t)\}$. Then we have

$$Y_t = L_t + \sum_{0<s\leq t} (e^{X_s - X_{s-}} - 1)(L_s - L_{s-}) + t\,\alpha. \tag{2.5}$$

If $\{X_t\}$ and $\{Y_t\}$ are independent, then $L_t = Y_t$. Lindner and Maller [23] proved the following.

Theorem 2.1. *Suppose that there is no constant k such that $M = k$ and $V_t = k$ satisfy* (2.3). *Then the generalized Ornstein–Uhlenbeck process $\{V_t\}$ induced by $\{(X_t, Y_t)\}$ has a stationary distribution if and only if*

$$\int_0^{\infty-} e^{-X_{s-}} dL_s \quad \textit{is convergent a. s.} \tag{2.6}$$

If (2.6) *is satisfied, then the distribution of the integral in* (2.6) *is a stationary distribution of $\{V_t\}$. A stationary distribution of $\{V_t\}$ is unique if it exists.*

The condition that $\int_0^{\infty-} e^{-X_{s-}} dY_s$ is convergent a. s. is a sufficient condition but not a necessary condition for (2.6). A necessary and sufficient condition for (2.6) in terms of the generating triplet of $\{(X_t, L_t)\}$ is found in Erickson and Maller [14]. If $\{L_t\}$ is not a zero process, then the condition $\lim_{t\to\infty} X_t = +\infty$ a. s. is necessary for (2.6). The expression of this condition in terms of the generating triplet of $\{X_t\}$ is given by Sato [45] and Doney and Maller [12]. If $E[|X_1|] < \infty$ and $E[X_1] > 0$, then (2.6) is equivalent to the condition $E[\log^+ |L_1|] < \infty$, which is also equivalent to $E[\log^+ |Y_1|] < \infty$, as [23] says.

Recalling that the class of stationary distributions of all Ornstein–Uhlenbeck processes is identical to the class of selfdecomposable distributions, we are interested in what properties the stationary distributions of generalized Ornstein–Uhlenbeck processes have. This problem is to study the distribution μ of the integral of the form

$$\int_0^{\infty-} e^{-X_{s-}} dL_s. \tag{2.7}$$

Bertoin, Lindner, and Maller [6] show that μ is continuous (that is, any one-point set has μ-measure 0) except in the case of one-point distributions. Further, it follows from Theorem 1.3 of the recent work [1] of Alsmeyer, Iksanov, and Rösler that μ is either absolutely continuous or continuous-singular except in the case of one-point distributions (this was pointed out by Toshiro Watanabe; A. Lindner also noticed the same fact by an extension of the method of Grintsevichyus [16]). In general, a measure μ is called absolutely continuous if every set of Lebesgue measure 0 is of μ-measure 0; μ is called singular if it is concentrated on a set of Lebesgue measure 0; μ is called continuous-singular if it is continuous and singular.

Example 2.2. Suppose that, for some $a > 0$, $X_t - at$ is a subordinator and that $L_t = t$. Then the distribution μ of (2.7) is not infinitely divisible except in the case of one-point distribution, because

$$\int_0^{\infty-} e^{-X_{s-}} ds = \int_0^{\infty-} e^{-(X_{s-}-as)-as} ds \le 1/a$$

and μ has a bounded support.

Example 2.3. Suppose that $\nu_X((0,\infty)) = 0$, that is, $\{X_t\}$ does not have positive jumps, and that (2.6) is valid. Then the distribution μ of (2.7) is selfdecomposable. The proof is as follows. Let a be an arbitrary positive number and let T_a be the hitting time of a for $\{X_t\}$. Except in the case that $\{L_t\}$ is a zero process, we have $T_a < \infty$ a. s. and

$$\int_0^{\infty-} e^{-X_{s-}} dL_s = \int_0^{T_a} e^{-X_{s-}} dL_s + e^{-a} \int_{T_a}^{\infty-} e^{-(X_{s-}-X_{T_a})} dL_s.$$

Since the homogeneous independent increment property of a Lévy process remains true for stopping times, the first and second terms in the right-hand side are independent and the distribution of the second term is the same as that of $e^{-a}\int_0^{\infty-} e^{-X_{s-}} dL_s$. Therefore, letting ρ denote the distribution of the first term, we have $\widehat{\mu}(z) = \widehat{\mu}(e^{-a}z)\widehat{\rho}(z)$ ([6, 20]).

2.2. **Poissonian exponential integrals.** Let $\{(N_t, L_t)\}$ be a Lévy process on $\mathbb{R}^2$ such that $\{N_t\}$ is a Poisson process. Let $c > 1$. Consider $(X_t, L_t) = ((\log c)N_t, L_t)$, a Lévy process on $\mathbb{R}^2$. Then (2.7) equals

$$\int_0^{\infty-} c^{-N_{s-}} dL_s. \tag{2.8}$$

We call (2.8) a *Poissonian exponential integral.*

Theorem 2.4. *If the integral* (2.8) *is convergent a. s., then its distribution* μ *is* c^{-1}*-decomposable.*

Proof. The first jump time T_1 for $\{N_t\}$ is the hitting time of $\log c$ for $\{X_t\}$. Hence the argument in Example 2.3 gives $\widehat{\mu}(z) = \widehat{\mu}(c^{-1}z)\widehat{\rho}(z)$, where ρ is the distribution of L_{T_1}. □

It follows from this theorem that $\widehat{\mu}(z) = \widehat{\mu}(c^{-n}z)\prod_{j=0}^{n-1}\widehat{\rho}(c^{-j}z)$ for every positive integer n and that

$$\widehat{\mu}(z) = \prod_{j=0}^{\infty}\widehat{\rho}(c^{-j}z). \tag{2.9}$$

In order for (2.8) to be convergent a. s., it is necessary and sufficient that $E[\log^+|L_1|] < \infty$. We give four examples of Poissonian exponential integrals under the assumption that $\{N_t\}$ and $\{L_t\}$ are independent.

Example 2.5. Suppose that (2.8) is convergent a. s. Then μ is infinitely divisible. Indeed, T_1 has exponential distribution (hence infinitely divisible distribution) and $\{L_t\}$ and T_1 are independent, from which follows the infinite divisibility of the distribution ρ of L_{T_1} by the theory of subordination. Hence μ is infinitely divisible from (2.9).

Example 2.6. Let $\{L_t\}$ be a strictly stable process with index α $(0 < \alpha \le 2)$ or a Brownian motion with drift. Then the integral (2.8) is convergent a. s. and μ is selfdecomposable. Indeed, an exponential distribution is selfdecomposable and hence L_{T_1} has selfdecomposable distribution by the result of Sato [46]. This gives the selfdecomposability of μ by (2.9).

Example 2.7. Let $L_t = t$. Then μ is selfdecomposable. This is treated by Bertoin, Biane, and Yor [5]. But, as is pointed out in [20], this is a special case of Example 2.6, since $\{L_t\}$ is strictly stable with index 1.

Example 2.8. Suppose that $\{L_t\}$ is also a Poisson process. Then μ is infinitely divisible by Example 2.5, but μ is not selfdecomposable, as will be explained in Subsection 2.5.

Now let us consider the following sequence of Poissonian exponential integrals. This sequence is interesting as we can completely analyze its infinite divisibility and make a deep study of its absolute continuity and continuous-singularity. Let u, v, and w be non-negative numbers satisfying $u + w > 0$ and $v + w > 0$ and let $c > 1$. For each $k \in \mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ let $\{(N_t^{(k)}, L_t^{(k)})\}$ be a Lévy process on $\mathbb{R}^2$ with characteristic function

$$E[e^{i(z_1 N_t^{(k)} + z_2 L_t^{(k)})}] = \exp\left[t\int_{\mathbb{R}^2}(e^{i(z_1x_1+z_2x_2)} - 1)\nu_{(N^{(k)},L^{(k)})}(dx)\right], \tag{2.10}$$

where the Lévy measure $\nu_{(N^{(k)},L^{(k)})}$ is concentrated at three points $(1,0)$, $(0,1)$, and $(1,c^{-k})$ with mass u, v, and w. Let p, q, and r be the normalization of u, v, and w; that is, $p=u/(u+v+w)$, $q=v/(u+v+w)$, and $r=w/(u+v+w)$. Thus p, q, and r are non-negative and $p+q+r=1$. Since $p+r>0$ and $q+r>0$ are assumed, we have $p<1$ and $q<1$. The Lévy process $\{(N_t^{(k)},L_t^{(k)})\}$ has a simple structure called a compound Poisson process. It moves only by jumps. Let $\widetilde{T}_1$ be the first jump time of $\{(N_t^{(k)},L_t^{(k)})\}$ and let $\widetilde{T}_2,\widetilde{T}_3,\dots$ be the waiting times for the consecutive jumps. Let $J_1,J_2,\dots$ be the sizes of the successive jumps. Then $\widetilde{T}_1,\widetilde{T}_2,\dots,J_1,J_2,\dots$ are independent and, for all j, $\widetilde{T}_j$ has exponential distribution with mean $1/(u+v+w)$ and the distribution of J_j is the normalization of $\nu_{(N^{(k)},L^{(k)})}$. It follows that

$$\{(N_t^{(k)},L_t^{(k)})\}\stackrel{\mathrm{d}}{=}\{(N_t^{(k-1)},\widetilde{L}_t^{(k)})\},$$

where we define

$$\widetilde{L}_t^{(k)}=L_t^{(k-1)}+\sum_{0<s\le t}(e^{-(\log c)(N_s^{(k-1)}-N_{s-}^{(k-1)})}-1)(L_s^{(k-1)}-L_{s-}^{(k-1)}).$$

Thus, letting $X_t^{(k)}=(\log c)N_t^{(k)}$, we see from Theorem 2.1 that the stationary distribution $\mu^{(k)}$ of the generalized Ornstein–Uhlenbeck process induced by $\{(X_t^{(k-1)},L_t^{(k-1)})\}$ is given by the distribution of

$$\int_0^{\infty-}c^{-N_{s-}^{(k)}}dL_s^{(k)}=\int_0^{\infty-}e^{-X_{s-}^{(k)}}dL_s^{(k)}. \tag{2.11}$$

The processes $\{N_t^{(k)}\}$ and $\{L_t^{(k)}\}$ are Lévy processes on $\mathbb{R}$ and their Lévy measures $\nu_{N^{(k)}}$ and $\nu_{L^{(k)}}$ are the projections of $\nu_{(N^{(k)},L^{(k)})}$ to the first and second axes with the mass at the point 0 deleted. Hence $\nu_{N^{(k)}}$ is supported on one point 1 with mass $u+w$; $\{N_t^{(k)}\}$ is a Poisson process with parameter $u+w$. Hence (2.11) is a Poissonian exponential integral. For $k\neq 0$, $\nu_{L^{(k)}}$ is supported on at most two points 1 and c^{-k} with mass v and w and hence, if $w>0$, $\{L_t^{(k)}\}$ is not a Poisson process. For $k=0$, $\nu_{L^{(0)}}$ is supported at 1 with mass $v+w$ and hence $\{L_t^{(0)}\}$ is a Poisson process with parameter $v+w$. The integral (2.11) is a pathwise Stieltjes integral and its convergence is seen from $N_t^{(k)}/t\to E(N_1^{(k)})$ and $L_t^{(k)}/t\to E(L_1^{(k)})$ a. s. as $t\to\infty$ by the strong law of large numbers for Lévy processes. Note that $w=0$ or $r=0$ is equivalent to the independence of $\{N_t^{(k)}\}$ and $\{L_t^{(k)}\}$.

In the following part of Section 2 we write about the properties of the sequence $\mu^{(k)}$, $k\in\mathbb{Z}$, of distributions of Poissonian exponential integrals. They are results of Lindner and Sato [24, 25]. Let T be the first jump time of $\{N_t^{(k)}\}$ and let $\rho^{(k)}$ be the distribution of $L_T^{(k)}$. The characteristic functions of $\mu^{(k)}$ and $\rho^{(k)}$ have explicit expressions.

Lemma 2.9. *For each $k\in\mathbb{Z}$ we have*

$$\rho^{(k)}=\sum_{m=0}^{\infty}q^m p\,\delta_m+\sum_{m=0}^{\infty}q^m r\,\delta_{m+c^{-k}},\qquad \widehat{\rho}^{(k)}(z)=\frac{p+re^{ic^{-k}z}}{1-qe^{iz}}, \tag{2.12}$$

$$\widehat{\mu}^{(k)}(z)=\prod_{n=0}^{\infty}\frac{p+re^{ic^{-k-n}z}}{1-qe^{ic^{-n}z}}, \tag{2.13}$$

$$\widehat{\mu}^{(k)}(z) = \widehat{\mu}^{(k+1)}(z)\frac{p + re^{ic^{-k}z}}{p+r}, \tag{2.14}$$

$$\widehat{\mu}^{(k+1)}(z) = \widehat{\mu}^{(k)}(c^{-1}z)\frac{1-q}{1-qe^{iz}}. \tag{2.15}$$

We have started from the given u, v, w, and c, but this lemma shows that, in order to study $\mu^{(k)}$ and $\rho^{(k)}$, we can consider that the parameters are p, q, r, and c satisfying $p + q + r = 1$.

Proof of Lemma 2.9. If $J_1 = (1, 0)$, then $L_T^{(k)} = 0$; if $J_1 = (1, c^{-k})$, then $L_T^{(k)} = c^{-k}$; if $J_1 = \cdots = J_m = (0, 1), J_{m+1} = (1, 0)$, then $L_T^{(k)} = m$; if $J_1 = \cdots = J_m = (0, 1), J_{m+1} = (1, c^{-k})$, then $L_T^{(k)} = m + c^{-k}$. Hence we obtain (2.12). Then an expression like (2.9) gives (2.13). Finally (2.14) and (2.15) follow from (2.13). □

2.3. **Properties of a sequence of Poissonian exponential integrals. Absolutely continuous or not.** Given p, q, r, and c, let $\mu^{(k)}$, $k \in \mathbb{Z}$, be the sequence of the distributions of the Poissonian exponential integrals defined in Subsection 2.2. By the following theorem or by a result of [1] mentioned in Subsection 2.1, $\mu^{(k)}$ is either absolutely continuous or continuous-singular.

Theorem 2.10 (Wolfe [61]). *Let μ be a distribution on $\mathbb{R}^d$ which is not a one-point distribution. If μ is b-decomposable for some $b \in (0, 1)$, then it is either absolutely continuous or continuous-singular.*

The infimum of the Hausdorff dimensions of B over all $B \in \mathcal{B}(\mathbb{R})$ satisfying $\mu(B) = 1$ is called the Hausdorff dimension of μ, denoted by $\dim(\mu)$. If $\dim(\mu) < 1$, then μ is singular.

Theorem 2.11. *Fix p, q, r, and c. Let $k \in \mathbb{Z}$. Then $\mu^{(k)}$ is absolutely continuous if and only if $\mu^{(0)}$ is absolutely continuous; $\mu^{(k)}$ is continuous-singular if and only if $\mu^{(0)}$ is continuous-singular. Further,* $\dim(\mu^{(k)}) = \dim(\mu^{(0)})$.

Indeed, notice that (2.14) can be written as

$$\mu^{(k)}(B) = \frac{p}{p+r}\mu^{(k+1)}(B) + \frac{r}{p+r}\mu^{(k+1)}(B - c^{-k}), \quad B \in \mathcal{B}(\mathbb{R}).$$

Thus the properties of $\mu^{(k)}$ and $\mu^{(k+1)}$ are related as asserted.

Now the problem is to distinguish the case where $\mu^{(0)}$ is absolutely continuous and the case where $\mu^{(0)}$ is continuous-singular. However this is very difficult. In relation to this, notice that if $q = 0$, then $\rho^{(0)}$ is a two-point distribution located at 0 and 1 and $\mu^{(0)}$ is called an *infinite Bernoulli convolution*; the question on absolute continuity and continuous-singularity of it is a long-standing hard problem since the 1930s (see [35] for a survey). As we will assume that $q > 0$, our results do not cover infinite Bernoulli convolutions but the concepts and techniques developed in the course of their investigation are helpful in the study of $\mu^{(0)}$. In the following we will give sufficient conditions for the absolute continuity of $\mu^{(0)}$ and for the continuous-singularity of $\mu^{(0)}$; the results heavily depend on the parameter values.

A number $c > 1$ is called a *Pisot–Vijayaraghavan number* (PV number) or a Pisot number if c is a root of a polynomial $F(x)$ with integer coefficients with 1 at the highest term and the other roots of $F(x)$ have absolute value < 1. The set of PV numbers is denoted by PV. It is a countable set. For example, all integers greater than 1 are in PV; the positive root $(1+\sqrt{5})/2 \approx 1.6180$ of $F(x) = x^2 - x - 1$

and the unique real root ≈ 1.3247 of $F(x) = x^3 - x - 1$ are in PV. According to [35], $(1+\sqrt{5})/2$ is the smallest cluster point of PV. In 1939 Erdős [13] first used PV numbers in the study of this subject.

Next, following Watanabe [59], we call $b \in (0,1)$ a *Peres–Solomyak number* (PS number) if there are $p_0 \in (1/2, 1)$ and a positive integer m such that $\widehat{\mu}_0(z)^m$ is integrable on $\mathbb{R}$ for the distribution μ_0 of $\sum_{n=0}^{\infty} b^n U_n$ (infinite Bernoulli convolution) with independent random variables $\{U_n\}$ distributed as $P(U_n = 0) = p_0$ and $P(U_n = 1) = 1 - p_0$. Let $(PS)^{-1}$ denote the set of reciprocals of PS numbers. As is explained in [59], the set $(1,\infty) \setminus (PS)^{-1}$ has Lebesgue measure 0 and $PV \cap (PS)^{-1} = \emptyset$, but it is not known whether $PV \cup (PS)^{-1} = (1,\infty)$ or not; no explicit PS number is known.

In the following part of this subsection we assume $q > 0$.

Theorem 2.12. *If $c \in PV$, then $\mu^{(0)}$ is continuous-singular.*

The proof is based on the idea of Erdős [13]. If the upper limit of $|\widehat{\mu}^{(0)}(z)|$ as $z \to \infty$ is positive, then $\mu^{(0)}$ is not absolutely continuous, by the Riemann–Lebesgue theorem. Starting from (2.13) of Lemma 2.9 and using the assumption that $c \in PV$, we can estimate $|\widehat{\mu}^{(0)}(z)|$ from below.

In general, for a discrete (that is, concentrated on a countable set C) distribution ρ, its entropy $H(\rho)$ is defined by $H(\rho) = -\sum_{a \in C} \rho(\{a\}) \log \rho(\{a\})$.

The following two theorems are known.

Theorem 2.13 (Watanabe [59])**.** *Suppose that b is a PS number. If μ is b-decomposable, then, for all large n, the n-fold convolution μ^{n*} of μ is absolutely continuous with bounded continuous density.*

Theorem 2.14 (Watanabe [59])**.** *Let b be an arbitrary number in $(0,1)$. If μ is b-decomposable and the factor ρ in (1.2) is discrete, then $\dim(\mu) \le H(\rho)/(-\log b)$.*

Using the method of proof of Theorem 2.13, we can show

Theorem 2.15. *Let $c \in (PS)^{-1}$. Then there exists $\varepsilon = \varepsilon(c) \in (0,1)$ such that, if $p > 0$, $r \le pq$, and $q \ge 1 - \varepsilon$, then $\mu^{(0)}$ is absolutely continuous with bounded continuous density.*

Theorem 2.14 is directly applied to $\mu^{(0)}$ and we have $\dim(\mu^{(0)}) \le H(\rho^{(0)})/\log c$. Since $\rho^{(0)}$ is determined by p, q, and r, we have

Corollary 2.16. *Fix p, q, and r. Then, for all sufficiently large c, $\mu^{(0)}$ is continuous-singular.*

It is a surprising fact that there exist countable special numbers for c as in Theorem 2.12 while $\dim(\mu^{(0)})$ has an upper estimate monotone in c.

We also have $\dim(\mu^{(0)}) \le H(\rho^{(k)})/\log c$ for $k \in \mathbb{Z}$ by Theorem 2.11. We can calculate $H(\rho^{(k)})$ from (2.12) and show that

$$H(\rho^{(k)}) \le H(\rho^{(1)}) = (-p \log p - q \log q - r \log r)/(p+r) \le (\log 3)/(p+r)$$

in general and

$$H(\rho^{(k)}) \le H(\rho^{(1)}) = (-p \log p - q \log q)/p \le (\log 2)/p \quad \text{if } r = 0.$$

Example 2.17. If $(\log 3)/((p+r)\log c) < 1$ or if, more strongly, $p + r = 1/2$ and $c > 9$, then $\mu^{(0)}$ is continuous-singular. In the case $r = 0$, if $(\log 2)/(p \log c) < 1$ or if, more strongly, $c > 4$ and $p = 1/2$ or $c = e$ and $p > \log 2 \approx 0.69315$, then $\mu^{(0)}$ is continuous-singular.

Using the theorems in Subsection 2.5, we will see that there are many cases where $\mu^{(k)}$ is continuous-singular and infinitely divisible. The existence of distributions with this property has been known since Rubin [39] and Tucker [55]. But it is remarkable that distributions of such a simple form as our $\mu^{(k)}$ are examples for it.

We do not have an example of absolutely continuous $\mu^{(0)}$ with explicitly given parameters.

For a general Lévy process $\{Y_t\}$, the distribution σ^{t*} of Y_t can have time evolution of a qualitative nature ([44, 60]). One such phenomenon is the existence of a critical time t_0 such that σ^{t*} is continuous-singular for $0 < t < t_0$ and absolutely continuous for $t > t_0$; examples of this are found in [44, 59]. We can show that, with an appropriate choice of p, q, r, and c, $\mu^{(k)}$ is in ID and serves as another example for this.

2.4. **Quasi-infinitely divisible distributions.** We say that a distribution μ on $\mathbb{R}$ is *quasi-infinitely divisible* if it has a cumulant function of the form (1.1) of Lévy–Khintchine representation for $d = 1$ with the assumption that A is real and ν is a signed measure satisfying $\nu(\{0\}) = 0$ and $\int (|x|^2 \wedge 1)|\nu|(dx) < \infty$, where $|\nu|$ is the total variation measure of ν. If μ is quasi-infinitely divisible, then A and ν are uniquely determined; ν is called the *quasi-Lévy measure* of μ. A quasi-infinitely divisible distribution μ is infinitely divisible if and only if $A \geq 0$ and the negative part in the Jordan decomposition of ν is zero. Such distributions are already considered in Gnedenko–Kolmogorov [15, p. 81], Linnik–Ostrovskiĭ [26, Chap. 6, § 7], Niedbalska-Rajba [34], and others. To find a necessary and sufficient condition for a signed measure to be a quasi-Lévy measure is an open problem. We do not have understanding of the role of quasi-infinitely divisible distributions in probability theory. This is to be studied.

In the rest of this section ID stands for $ID(\mathbb{R}^d)$, ID^0 is the class of quasi-infinitely divisible distributions on $\mathbb{R}$ which are not infinitely divisible, and ID^{00} is the class of distributions on $\mathbb{R}$ which are not quasi-infinitely divisible.

Example 2.18. Let $\rho = p\delta_0 + r\delta_1$, a two-point distribution on $\mathbb{R}$ with $0 < p < 1$ and $r = 1 - p$. We have $\widehat{\rho}(z) = p + re^{iz}$. (i) If $p = r$, then $\rho \in ID^{00}$. (ii) If $p \neq r$, then $\rho \in ID^0$. Indeed, (i) is true because $\widehat{\rho}(\pi) = 0$ and ρ does not have a cumulant function. (ii) is seen from the fact that

$$\widehat{\rho}(z) = \begin{cases} \exp\left[\sum_{m=1}^{\infty} (-1)^{m+1} m^{-1} (r/p)^m (e^{imz} - 1)\right] & \text{for } p > r, \\ \exp\left[\sum_{m=1}^{\infty} (-1)^{m+1} m^{-1} (p/r)^m (e^{-imz} - 1) + iz\right] & \text{for } p < r. \end{cases}$$

Notice that if $p < r$, then ρ is a distribution on $\mathbb{R}_+$ but its quasi-Lévy measure is on $(-\infty, 0)$.

2.5. **Properties of a sequence of Poissonian exponential integrals. Infinitely divisible or not. Quasi-infinitely divisible or not.** As in the preceding subsections, for given p, q, r, and c, let $\mu^{(k)}$, $k \in \mathbb{Z}$, be the sequence of the distributions of the Poissonian exponential integrals defined in Subsection 2.2. We are assuming that $p, q, r \geq 0$, $c > 1$, $p + q + r = 1$, $p + r > 0$, and $q + r > 0$ throughout. We will give criteria for $\mu^{(k)}$ and $\rho^{(k)}$ to belong to ID, ID^0, or ID^{00} in the form of explicit conditions for p, q, r, and c. First we show some properties satisfied by $\mu^{(k)}$ and $\rho^{(k)}$ for all $k \in \mathbb{Z}$.

Theorem 2.19. (i) *If* $p = 0$ *or* $r = 0$, *then* $\rho^{(k)}, \mu^{(k)} \in ID$. (ii) *If* $0 < r < p$, *then* $\rho^{(k)}, \mu^{(k)} \in ID \cup ID^0$. (iii) *If* $0 < p < r$, *then* $\rho^{(k)}, \mu^{(k)} \in ID^0$. (iv) *If* $p = r > 0$, *then* $\rho^{(k)}, \mu^{(k)} \in ID^{00}$.

Proof. (i) Use (2.12). If $r = 0$, then $\rho^{(k)}$ is a geometric distribution. If $p = 0$, then $\rho^{(k)}$ is a translation of a geometric distribution. Thus $\rho^{(k)} \in ID$ in both cases. Hence also $\mu^{(k)} \in ID$. (ii) Assume $0 < r < p$. It follows from (2.12) that $\widehat{\rho}^{(k)}(z)$ equals the right-hand side of (1.6) with $\gamma_0 = 0$ and ν a finite signed measure

$$\nu_{\rho^{(k)}} = \sum_{m=1}^{\infty} \left[m^{-1} q^m \delta_m + (-1)^{m+1} m^{-1} (r/p)^m \delta_{mc^{-k}} \right]. \tag{2.16}$$

Hence $\rho^{(k)} \in ID \cup ID^0$. Then it follows from (2.13) that $\widehat{\mu}^{(k)}(z)$ also equals the right-hand side of (1.6) with $\gamma_0 = 0$ and where ν is

$$\nu_{\mu^{(k)}} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[m^{-1} q^m \delta_{mc^{-n}} + (-1)^{m+1} m^{-1} (r/p)^m \delta_{mc^{-k-n}} \right], \tag{2.17}$$

and $\int_{(0,\infty)} x \, |\nu_{\mu^{(k)}}|(dx) < \infty$. This means $\mu^{(k)} \in ID \cup ID^0$. (iii) Assume $0 < p < r$. We can write (2.12) as $\widehat{\rho}^{(k)}(z) = (1 + (p/r) e^{-ic^{-k} z})(1 - q e^{iz})^{-1} r e^{ic^{-k} z}$ and show that $\widehat{\rho}^{(k)}(z)$ has the form of the right-hand side of (1.6) with $\gamma_0 = c^{-k}$ and where ν is

$$\nu_{\rho^{(k)}} = \sum_{m=1}^{\infty} \left[m^{-1} q^m \delta_m + (-1)^{m+1} m^{-1} (p/r)^m \delta_{-mc^{-k}} \right].$$

As the negative part of $\nu_{\rho^{(k)}}$ is not zero, we see that $\rho^{(k)} \in ID^0$. By (2.13), $\widehat{\mu}^{(k)}(z)$ also equals the right-hand side of (1.6) with $\gamma_0 = c^{1-k}(c-1)^{-1}$ and ν obtained from $\nu_{\rho^{(k)}}$ similarly to (2.17). It follows that $\int |x| \, |\nu_{\mu^{(k)}}|(dx) < \infty$ and hence $\mu^{(k)} \in ID \cup ID^0$. If $\mu^{(k)} \in ID$, then $\nu_{\mu^{(k)}}$ is supported on the positive axis since $\mu^{(k)}$ is supported on the positive axis. However, since

$$\int_{(-\infty,0)} |x| \, \nu_{\mu^{(k)}}(dx) = c^{1-k}(c-1)^{-1} p (p+r)^{-1} \neq 0,$$

we conclude that $\mu^{(k)} \in ID^0$. (iv) If $p = r > 0$, then neither $\rho^{(k)}$ nor $\mu^{(k)}$ has a cumulant function, as their characteristic functions have zeros. □

Whether $\mu^{(k)}$ is absolutely continuous or continuous-singular does not depend on k, as is shown in Subsection 2.3, but whether $\mu^{(k)}$ is infinitely divisible or not does depend on k. First, $\mu^{(k)}$ has a kind of monotonicity in the following sense.

Theorem 2.20. *Let* $k \in \mathbb{Z}$. *If* $\mu^{(k)} \in ID$, *then* $\mu^{(k+1)} \in ID$. *If* $\mu^{(k)} \in ID \cup ID^0$, *then* $\mu^{(k+1)} \in ID \cup ID^0$.

This is because we have (2.15) and $(1-q)/(1-qe^{iz})$ is the characteristic function of a geometric distribution.

We see from Theorem 2.19 that the classification of $\rho^{(k)}$ and $\mu^{(k)}$ remains only in the case $0 < r < p$. The basis of our results below is the representations (2.16) and (2.17) of the quasi-Lévy measures $\nu_{\rho^{(k)}}$ and $\nu_{\mu^{(k)}}$. Since they are discrete, $\rho^{(k)}$ or $\mu^{(k)}$ is in ID^0 if and only if, for some point a, $\nu_{\rho^{(k)}}(\{a\}) < 0$ or $\nu_{\mu^{(k)}}(\{a\}) < 0$, respectively. Careful inspection is needed, as some terms in (2.16) or (2.17) correspond to the same point a. We formulate our results without proofs and give several remarks.

Theorem 2.21. *If $0 < r < p$, then there exists $k_0 \in \mathbb{Z}$ such that $\mu^{(k)} \in ID^0$ for all $k < k_0$.*

The following theorems are for $k = 0$, $k > 0$, and $k < 0$ in turn.

Theorem 2.22. *Let $0 < r < p$.* (i) *If $r \le pq$, then $\rho^{(0)}$ and $\mu^{(0)}$ are in ID.* (ii) *If $r > pq$, then $\rho^{(0)}$ and $\mu^{(0)}$ are in ID^0.*

Let $a_m = m^{-1}(q^m - (-r/p)^m)$. Since we have $\nu_{\rho^{(0)}} = \sum_{m=1}^{\infty} a_m \delta_m$ and $\nu_{\mu^{(0)}} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_m \delta_{mc^{-n}}$, (i) of Theorem 2.22 and the assertion on $\rho^{(0)}$ in (ii) are easily proved. Notice that, by (2.12), $\rho^{(0)}$ is the normalization of a geometric distribution with some mass at the origin deleted; Theorem 2.22 decides when it is infinitely divisible.

Example 2.23. Assume $2p = q$. Then $\rho^{(0)}$ and $\mu^{(0)}$ are in ID if and only if $r \le (13 - 3\sqrt{17})/4 \approx 0.15767$.

Theorem 2.24. *Assume that $0 < r < p$ and k is a positive integer.* (i) *$\rho^{(k)} \in ID$ if and only if $c^k = 2$ and $r^2 \le p^2 q$.* (ii) *$\mu^{(k)} \in ID$ if and only if one of the following is true:* (a) *$r \le pq$;* (b) *$r^2 \le p^2 q$ and $c^l = 2$ for some $l \in \{1, 2, \dots, k\}$.*

If $k > 0$, then $\nu_{\mu^{(k)}} = \sum_{n=k}^{\infty} \sum_{m=1}^{\infty} a_m \delta_{mc^{-n}} + \sum_{n=0}^{k-1} \sum_{m=1}^{\infty} m^{-1} q^m \delta_{mc^{-n}}$.

Corollary 2.25. *Let $0 < r < p$. Then $\mu^{(k)} \in ID^0$ for all $k \in \mathbb{Z}$ if and only if one of the following is true:* (a) *$r^2 > p^2 q$;* (b) *$p^2 q^2 < r^2 \le p^2 q$ and $c^l \ne 2$ for every positive integer l.*

Theorem 2.26. *Assume that $0 < r < p$ and k is a negative integer.* (i) *Let $\alpha = 2c^{|k|}$. Then $\rho^{(k)} \in ID$ if and only if α is an integer and $q^{\alpha} \ge (\alpha/2)(r/p)^2$.* (ii) *If $2c^j$ is not an integer for any integer $j \ge |k|$, then $\mu^{(k)} \in ID^0$.*

If $k < 0$, then $\nu_{\mu^{(k)}} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_m \delta_{c^{-n}m} - \sum_{j=1}^{|k|} \sum_{m=1}^{\infty} m^{-1}(-r/p)^m \delta_{c^j m}$. For an integer $\alpha \ge 2$ let

$$F_{\alpha}(x) = \sum_{n=0}^{\infty} \alpha^{-n} x^{2\alpha^n}, \quad 0 \le x \le 1,$$

and let $f_{\alpha}(x)$, $0 < x \le 1$, be the function uniquely defined by $\alpha^{-1} F_{\alpha}(x) = F_{\alpha}(f_{\alpha}(x)x)$. The next lemma is essential for showing Theorem 2.28.

Lemma 2.27. *The function $f_{\alpha}(x)$ satisfies the following.* (i) *It is continuous and strictly increasing for $0 < x \le 1$.* (ii) *$f_{\alpha}(x) \to \alpha^{-1/2}$ as $x \downarrow 0$.* (iii) *$f_{\alpha}(1) < \alpha^{-1/4}$.* (iv) *$f_{\alpha}(1) < \alpha^{-1/2}(1 + \alpha^{-1})$ for all sufficiently large α.* (v) *$f_{\alpha}(x^n) \ge (f_{\alpha}(x))^n$ for every positive integer n.*

Further, for any integer $\alpha \ge 2$ and positive integer j, let $h_{\alpha}^j(x)$, $0 < x \le 1$, be the function uniquely defined by $\alpha^{-j} F_{\alpha}(x) = F_{\alpha}(h_{\alpha}^j(x))$. We have $h_{\alpha}^1(x) = f_{\alpha}(x)x$ and $h_{\alpha}^{j+1} = h_{\alpha}^1(h_{\alpha}^j(x))$.

Theorem 2.28. *Assume that $0 < r < p$ and k is a negative integer. Suppose that c^j is an integer for some positive integer j and let l be the smallest such j. Write $\alpha = c^l$. Then $\mu^{(k)} \in ID$ if and only if $q > 0$ and $h_{\alpha}^{\lceil |k|/l \rceil}(q^{\alpha^{\lceil |k|/l \rceil}}) \ge r/p$.*

Here $\lceil x \rceil$ denotes the smallest integer n satisfying $n \ge x$.

Theorem 2.29. *Assume that $0 < r < p$ and k is a negative integer. Suppose that $2c^j$ is an odd integer for some integer j satisfying $j \geq |k|$. Then $\mu^{(k)} \in ID$ if and only if $\mu^{(-1)} \in ID$.*

Now, what remains after the theorems above is to find a necessary and sufficient condition for $\mu^{(-1)} \in ID$ under the assumptions that $0 < r < p$ and $2c^j$ is an odd integer for some positive integer j. We can express the condition by a finite number of inequalities. But its formulation is somewhat complicated, which we omit in this article. In our result the number of inequalities that we use is 149.

We can see that the smaller r is, the more likely it is that $\rho^{(k)}$ and $\mu^{(k)}$ are infinitely divisible. We cannot conjecture the theorems in this subsection beforehand; they are conclusions obtained by trial and error. What hidden structure makes the infinite divisibility of $\mu^{(k)}$ appear or vanish? There is no clue to help with the answer.

From our discussion we see that, whenever $\mu^{(k)} \in ID$, $\mu^{(k)}$ is not selfdecomposable, since its Lévy measure is discrete.

As an application of Theorem 2.24, if $k > 0$, then we can find a non-empty domain of parameter values p, q, r, c such that $\mu^{(k)} \in ID$ and $\rho^{(k)} \notin ID$. This is the case if $0 < r \leq pq$ and $c^k \neq 2$. In general a b-decomposable infinitely divisible distribution μ can have a non-infinitely divisible factor ρ in (1.2), as Niedbalska-Rajba [34] shows by an elaborate example. Ours are the first such examples connected with stochastic processes.

2.6. **Symmetrization of a sequence of Poissonian exponential integrals.** The symmetrization σ^{sym} of a distribution σ is the distribution that has characteristic function $|\widehat{\sigma}(z)|^2$. Hence we can prove

Lemma 2.30. *If σ is a distribution on $\mathbb{R}_+$ in ID^0 and its quasi-Lévy measure is concentrated on $(0,\infty)$, then $\sigma^{\mathrm{sym}} \in ID^0$.*

Lemma 2.31. *Let $k \in \mathbb{Z}$. Let $(p', q', r', c') = (r, q, p, c)$ and let $\rho'^{(k)}$ and $\mu'^{(k)}$ be the distributions corresponding to (p', q', r', c'). Then $\rho'^{(k)\,\mathrm{sym}} = \rho^{(k)\,\mathrm{sym}}$ and $\mu'^{(k)\,\mathrm{sym}} = \mu^{(k)\,\mathrm{sym}}$.*

Using these two lemmas, the theorems in Subsection 2.5, and their proofs, we obtain

Theorem 2.32. *Let $k \in \mathbb{Z}$.* (i) *If $p = 0$ or $r = 0$, then $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ are in ID.* (ii) *If $p \neq r$, then $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ are in $ID \cup ID^0$.* (iii) *If $p = r > 0$, then $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ are in ID^{00}.*

Theorem 2.33. *Assume $p > 0$, $r > 0$, and $p \neq r$.* (i) *If $(r/p) \wedge (p/r) \leq q$, then $\rho^{(0)\,\mathrm{sym}}$ and $\mu^{(0)\,\mathrm{sym}}$ are in ID.* (ii) *If $(r/p) \wedge (p/r) > q$, then $\rho^{(0)\,\mathrm{sym}}$ and $\mu^{(0)\,\mathrm{sym}}$ are in ID^0.*

Example 2.34. Assume $2p = q$. Then $\rho^{(0)\,\mathrm{sym}}$ and $\mu^{(0)\,\mathrm{sym}}$ are in ID if and only if either $r \leq (13 - 3\sqrt{17})/4$ or $r \geq 1/2$. Compare this with Example 2.23.

Theorem 2.35. *Assume $p > 0$, $r > 0$, and $p \neq r$. Let k be a positive integer.* (i) *We have $\rho^{(k)\,\mathrm{sym}} \in ID$ if and only if $c^k = 2$ and $(r/p)^2 \wedge (p/r)^2 \leq q$.* (ii) *We have $\mu^{(k)\,\mathrm{sym}} \in ID$ if and only if one of the following is true:* (a) *$(r/p) \wedge (p/r) \leq q$;* (b) *$(r/p)^2 \wedge (p/r)^2 \leq q$ and $c^l = 2$ for some $l \in \{1, 2, \ldots, k\}$.*

We do not write our results in the case where k is a negative integer.

Let us give an application of Theorem 2.35. If $k > 0$, then it can happen that $\mu^{(k)\mathrm{sym}} \in ID$ and $\rho^{(k)\mathrm{sym}} \notin ID$. This is the case if $(r/p) \wedge (p/r) \leq q$ and $c^k \neq 2$. The distribution $\mu^{(k)\,\mathrm{sym}}$ is also c^{-1}-decomposable. Thus we see the existence of a symmetric infinitely divisible b-decomposable distribution μ for which the factor ρ in (1.2) is not infinitely divisible.

It is pointed out in [15] that some non-infinitely divisible distribution σ has infinitely divisible symmetrization σ^{sym}. We obtain many examples for this fact using Theorems 2.22, 2.24, 2.33, and 2.35.

3. Transformation of infinitely divisible distributions via stochastic integrals with respect to Lévy processes

3.1. Stochastic integrals on finite intervals. Let $f(s)$ be an extended real-valued measurable function on $[0,\infty)$. For $\mu \in ID(\mathbb{R}^d)$ let $X^{(\mu)} = \{X_t^{(\mu)} \colon t \geq 0\}$ denote a Lévy process on $\mathbb{R}^d$ such that $X_1^{(\mu)}$ has distribution μ. We say that f is *locally $X^{(\mu)}$-integrable* (on $[0,\infty)$) if $\int_0^t f(s)dX_s^{(\mu)}$ is definable for all $t \in [0,\infty)$. In order to obtain the condition for this, it is convenient to consider $\int_B f(s)dX_s^{(\mu)}$ for bounded Borel sets B as an independently scattered random measure, as Urbanik and Woyczyński [58] did employing the Nikodým theorem (see [21, 36, 47, 48]). Thus we have the following results. Let $\mathcal{B}_0(\mathbb{R}^d)$ be the class of Borel sets in $\mathbb{R}^d$ satisfying $\inf_{x\in B}|x| > 0$. Let (A,ν,γ) be the generating triplet of μ throughout Section 3.

Theorem 3.1. *Define*

$$\varphi_\mu(u) = u^2\mathrm{tr}A + \int_{\mathbb{R}^d}(|ux|^2 \wedge 1)\nu(dx) + \left| u\gamma + u\int_{\mathbb{R}^d} x\left(\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right|$$

for μ. (i) *A function $f(s)$ is locally $X^{(\mu)}$-integrable if and only if $\int_0^t \varphi_\mu(f(s))ds < \infty$ for $t \in [0,\infty)$.* (ii) *If $f(s)$ is locally $X^{(\mu)}$-integrable, then $\int_0^t |C_\mu(f(s)z)|ds < \infty$ for $z \in \mathbb{R}^d$ and $\int_0^t f(s)dX_s^{(\mu)}$ has distribution μ_t in $ID(\mathbb{R}^d)$ with $C_{\mu_t}(z) = \int_0^t C_\mu(f(s)z)ds$. The triplet (A_t,ν_t,γ_t) of μ_t is*

$$A_t = \int_0^t f(s)^2 A ds,$$

$$\nu_t(B) = \int_0^t ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx), \quad B \in \mathcal{B}_0(\mathbb{R}^d),$$

$$\gamma_t = \int_0^t f(s)ds\left(\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right).$$

Theorem 3.2. *A function $f(s)$ is locally $X^{(\mu)}$-integrable for all $\mu \in ID(\mathbb{R}^d)$ if and only if $f(s)$ is locally square-integrable, that is, $\int_0^t f(s)^2 ds < \infty$ for $t \in [0,\infty)$.*

The results above generalize to $\mathbb{R}^d$-valued additive processes and matrix-valued functions $f(s)$ ([48]).

3.2. Stochastic integrals on an infinite interval. In this subsection assume that $f(s)$ is locally square-integrable. If $\int_0^t f(s)dX_s^{(\mu)}$ is convergent a. s. as $t \to \infty$, the limit is called an *improper stochastic integral* and is written as $\int_0^{\infty-} f(s)dX_s^{(\mu)}$.

Its distribution is denoted by $\Phi_f(\mu)$. In this case we say that $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ (or $\Phi_f(\mu)$) is *definable*. The class of $\mu \in ID(\mathbb{R}^d)$ for which $\Phi_f(\mu)$ is definable is denoted by $\mathfrak{D}(\Phi_f)$ (with the dimension d fixed); this is the domain of the mapping Φ_f.

We use the following variants of stochastic integrals on the infinite interval $[0,\infty)$ (see also [11]). If, for a given $\mu \in ID(\mathbb{R}^d)$, there is a non-random $\mathbb{R}^d$-valued function $q(t)$ [resp. non-random point $q \in \mathbb{R}^d$] such that $\int_0^t f(s)dX_s^{(\mu)} - q(t)$ [resp. $\int_0^t f(s)dX_s^{(\mu*\delta_{-q})}$] is convergent a. s. as $t \to \infty$, we say that the limit stochastic integral of f with respect to $X^{(\mu)}$ is *essentially definable* [resp. *compensatedly definable*]; the class of all such μ is denoted by $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ [resp. $\mathfrak{D}(\Phi_{f,\mathrm{c}})$] and, for $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$ [resp. $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{c}})$], the class of the distributions of $\lim_{t\to\infty} \int_0^t f(s)dX_s^{(\mu)} - q(t)$ [resp. $\lim_{t\to\infty} \int_0^t f(s)dX_s^{(\mu*\delta_{-q})}$] where $q(t)$ [resp. q] ranges over all functions [resp. points] that can be chosen is denoted by $\Phi_{f,\mathrm{es}}(\mu)$ [resp. $\Phi_{f,\mathrm{c}}(\mu)$].

Theorem 3.3. (i) *In order that $\mu \in \mathfrak{D}(\Phi_f)$, it is necessary and sufficient that $\int_0^t C_\mu(f(s)z)ds$ is convergent for all $z \in \mathbb{R}^d$ as $t \to \infty$ or, equivalently, that the triplet (A,ν,γ) of μ satisfies*

$$\int_0^\infty f(s)^2 \mathrm{tr} A ds < \infty, \tag{3.1}$$

$$\int_0^\infty ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu(dx) < \infty, \tag{3.2}$$

$$\gamma_t \textit{ of Theorem } 3.1 \textit{ is convergent as } t \to \infty. \tag{3.3}$$

(ii) *If $\mu \in \mathfrak{D}(\Phi_f)$, then $\Phi_f(\mu) \in ID(\mathbb{R}^d)$, $C_{\Phi_f(\mu)}(z) = \int_0^{\infty-} C_\mu(f(s)z)ds$, and the triplet $(A_\infty,\nu_\infty,\gamma_\infty)$ of $\Phi_f(\mu)$ is*

$$A_\infty = \int_0^\infty f(s)^2 A ds,$$

$$\nu_\infty(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx), \quad B \in \mathcal{B}_0(\mathbb{R}^d),$$

and $\gamma_\infty = \lim_{t\to\infty} \gamma_t$.

We say that the improper stochastic integral of f with respect to $X^{(\mu)}$ is *absolutely definable* if μ satisfies $\int_0^\infty |C_\mu(f(s)z)|ds < \infty$ for $z \in \mathbb{R}^d$; the class of all such μ is denoted by $\mathfrak{D}^0(\Phi_f)$. Clearly

$$\mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_f) \subset \mathfrak{D}(\Phi_{f,\mathrm{c}}) \subset \mathfrak{D}(\Phi_{f,\mathrm{es}}). \tag{3.4}$$

Among these four classes $\mathfrak{D}^0(\Phi_f)$ and $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ are relatively easy to handle.

Theorem 3.4. *In order that $\mu \in \mathfrak{D}^0(\Phi_f)$, it is necessary and sufficient that (A,ν,γ) satisfies* (3.1), (3.2), *and*

$$\int_0^\infty \left| f(s)\left(\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\right| ds < \infty. \tag{3.5}$$

Theorem 3.5. *In order that $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$, it is necessary and sufficient that (A,ν,γ) satisfies* (3.1) *and* (3.2). *If $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$, then $\Phi_{f,\mathrm{es}}(\mu)$ is the class of all $\widetilde{\mu} \in ID(\mathbb{R}^d)$ having triplet $(A_\infty,\nu_\infty,\widetilde{\gamma})$, where A_∞ and ν_∞ are those in Theorem* 3.3 *and $\widetilde{\gamma} \in \mathbb{R}^d$.*

Theorem 3.6. *Assume* $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{c}})$. (i) *If* $\int_0^t f(s)ds$ *converges to a non-zero real number as* $t \to \infty$, *then* $\Phi_{f,\mathrm{c}}(\mu) = \Phi_{f,\mathrm{es}}(\mu)$. (ii) *If* $\int_0^t f(s)ds$ *is not convergent or if it converges to* 0 *as* $t \to \infty$, *then* $\Phi_{f,\mathrm{c}}(\mu)$ *is a singleton* $\{\widetilde{\mu}\}$ *in* $ID(\mathbb{R}^d)$ *and* $\widetilde{\mu}$ *has triplet* $(A_\infty, \nu_\infty, \widetilde{\gamma})$, *where* A_∞ *and* ν_∞ *are those in Theorem* 3.3 *and* $\widetilde{\gamma}$ *is some point. If, in addition,* $\int_{|x|>1} |x|\nu_\infty(dx) < \infty$, *then* $\widetilde{\gamma}$ *is such that* $\int_{\mathbb{R}^d} x\widetilde{\mu}(dx) = 0$.

In (ii), there is a case satisfying $\int_{|x|>1} |x|\nu_\infty(dx) = \infty$ even if $\int |x|\mu(dx) < \infty$.

It is an interesting problem how behaviors of $f(s)$ for large s influence the class $\mathfrak{D}(\Phi_f)$ and its variants. We say that $f(s) \asymp g(s)$, $s \to \infty$, if there are positive numbers c_1, c_2 such that, for all sufficiently large s, $0 < c_1 g(s) \le f(s) \le c_2 g(s)$. The transformation Φ_f for f in the following theorem is related to stable distributions with index α. The case $\alpha = 1$ deserves special attention, which reminds us of the fact that interesting phenomena on stable processes often emerge for $\alpha = 1$. For $\alpha > 0$, $\int |x|^\alpha \mu(dx) < \infty$ and $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ are equivalent.

Theorem 3.7. *Let* $\alpha > 0$. *Assume that* $f(s) \asymp s^{-1/\alpha}$, $s \to \infty$. (i) *If* $\alpha < 2$, *then* $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$ *if and only if* $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$. (ii) *If* $\alpha < 1$, *then*

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_{f,\mathrm{c}}) = \mathfrak{D}(\Phi_{f,\mathrm{es}}). \tag{3.6}$$

(iii) *Suppose that* $\alpha = 1$ *and* $\int_{s_0}^\infty |f(s) - cs^{-1}|ds < \infty$ *for some* $s_0 > 0$ *and* $c > 0$. *Then*

$$\mathfrak{D}^0(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{c}}) \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{es}});$$

$\mu \in \mathfrak{D}(\Phi_{f,\mathrm{c}})$ *if and only if* $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$ *and* $\int_{s_0}^t s^{-1}ds \int_{|x|>s} x\nu(dx)$ *is convergent as* $t \to \infty$; $\mu \in \mathfrak{D}(\Phi_f)$ *if and only if* $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{c}})$ *and* $\int x\mu(dx) = 0$; $\mu \in \mathfrak{D}^0(\Phi_f)$ *if and only if* $\mu \in \mathfrak{D}(\Phi_f)$ *and* $\int_{s_0}^\infty s^{-1}ds \left|\int_{|x|>s} x\nu(dx)\right| < \infty$. (iv) *Let* $1 < \alpha < 2$. *Then*

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{c}}) = \mathfrak{D}(\Phi_{f,\mathrm{es}});$$

$\mu \in \mathfrak{D}(\Phi_f)$ *if and only if* $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$ *and* $\int x\mu(dx) = 0$. (v) *If* $\alpha \ge 2$, *then*

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) = \{\delta_0\} \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{c}}) = \mathfrak{D}(\Phi_{f,\mathrm{es}}) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}.$$

A function f satisfying the condition in Theorem 3.8 below appears in (2.2) as well as in Subsection 3.4.

Theorem 3.8. *Let* $\alpha > 0$. *Assume that* $-\log f(s) \asymp s^\alpha$, $s \to \infty$. *Then* (3.6) *is true. We have* $\mu \in \mathfrak{D}(\Phi_f)$ *if and only if* $\int (\log^+ |x|)^{1/\alpha} \nu(dx) < \infty$, *that is,* $\int (\log^+ |x|)^{1/\alpha} \mu(dx) < \infty$.

Theorem 3.9. *Assume that* $f(s) \asymp e^{-e^s}$, $s \to \infty$. *Then* (3.6) *is true. We have* $\mu \in \mathfrak{D}(\Phi_f)$ *if and only if* $\int_{|x|>e} \log\log |x|\nu(dx) < \infty$, *that is,* $\int_{|x|>e} \log\log |x|\mu(dx) < \infty$.

We do not know in what context the function f in Theorem 3.9 appears. We also have

Theorem 3.10. *The following three statements are equivalent:* $\mathfrak{D}^0(\Phi_f) = ID(\mathbb{R}^d)$; $\mathfrak{D}(\Phi_{f,\mathrm{es}}) = ID(\mathbb{R}^d)$; $\int_0^\infty (f(s)^2 + 1_{\{f(s)\neq 0\}}(s))ds < \infty$.

The foregoing results in this section are given in [48, 49, 51]. Barndorff-Nielsen, Rosiński, and Thorbjørnsen [3] is an important related paper. Further, whether $\mathfrak{D}(\Phi_f)$ and its variants have monotonicity with respect to $|f|$ is treated in [50].

3.3. **Examples of stochastic integrals on an infinite interval.** First we introduce two subclasses of $ID(\mathbb{R}^d)$. Distributions μ_n are said to converge weakly to a distribution μ as $n \to \infty$ if $\int h(x)\mu_n(dx) \to \int h(x)\mu(dx)$ for all bounded continuous functions $h(x)$.

We call Ua an elementary Γ-variable in $\mathbb{R}^d$ if U is a real random variable having Γ distribution and $a \in \mathbb{R}^d \setminus \{0\}$. The *Thorin class* $T(\mathbb{R}^d)$ is defined to be the smallest class closed under convolution and weak convergence and containing the distributions of all elementary Γ-variables in $\mathbb{R}^d$. This was introduced by Thorin on $\mathbb{R}_+$ in the proof of the infinite divisibility of Pareto and log-normal distributions.

Goldie proved the infinite divisibility of mixtures of exponential distributions and Steutel found their Lévy measures. We call Va an elementary mixed-exponential variable in $\mathbb{R}^d$ if V is a real random variable whose law is a mixture of exponential distributions and $a \in \mathbb{R}^d \setminus \{0\}$. The *Goldie–Steutel–Bondesson class* $B(\mathbb{R}^d)$ is the smallest class closed under convolution and weak convergence and containing the distributions of all elementary mixed-exponential variables in $\mathbb{R}^d$. Bondesson [7] and Steutel and van Harn [53] give detailed exposition of these two classes on $\mathbb{R}_+$.

Now let us consider the domain $\mathfrak{D}(\Phi_f)$ and the range $\mathfrak{R}(\Phi_f) = \{\Phi_f(\mu) \colon \mu \in \mathfrak{D}(\Phi_f)\}$ of Φ_f for some specific functions f. We are interested in how the form of f is reflected in $\mathfrak{R}(\Phi_f)$.

Let $\alpha \in \mathbb{R}$, $g_\alpha(t) = \int_t^\infty u^{-\alpha-1}e^{-u}du$ for $t > 0$, and $b_\alpha = g_\alpha(0+)$. We have $b_\alpha = \Gamma(-\alpha)$ for $\alpha < 0$ and $b_\alpha = \infty$ for $\alpha \geq 0$. Let $t = f_\alpha(s)$, $0 < s < b_\alpha$, be the inverse function of $s = g_\alpha(t)$, $0 < t < \infty$. We define $f_\alpha(s) = 0$ for $s \notin (0, b_\alpha)$. Then, $f_\alpha(s) \sim \log(1/s)$ as $s \downarrow 0$. The behavior for $s \to \infty$ is that $f_0(s) \sim e^{-s}$ for $\alpha = 0$ and $f_\alpha(s) \sim s^{-1/\alpha}$ for $\alpha > 0$, ignoring non-zero constant multiples. If $\alpha = 1$, then the assumption in (iii) of Theorem 3.7 is satisfied. Write Ψ_α for Φ_{f_α}. For $\alpha < 0$ we have $\mathfrak{D}^0(\Psi_\alpha) = ID(\mathbb{R}^d)$. For $\alpha \geq 0$, $\mathfrak{D}(\Psi_\alpha)$ and its variants are described by using Theorems 3.7 and 3.8. If $\alpha = -1$, then $b_{-1} = 1$, $f_{-1}(s) = \log(1/s)$, and Ψ_{-1} is the mapping Υ introduced by Barndorff-Nielsen and Thorbjørnsen [4] in relation to free probability theory.

We use a decomposition of measures on $\mathbb{R}^d$ similar to that of Lévy measures of stable and selfdecomposable distributions in Section 1:

Lemma 3.11. *Let ρ be a σ-finite measure on $\mathbb{R}^d$ with $\rho(\{0\}) = 0$. Then there are a measure λ on the unit sphere S and a system $\{\rho_\xi \colon \xi \in S\}$ of measures on $(0,\infty)$ such that $0 \leq \lambda(S) \leq \infty$, $0 < \rho_\xi((0,\infty)) \leq \infty$, ρ_ξ is measurable in ξ, and*

$$\rho(B) = \int_S \lambda(d\xi) \int_{(0,\infty)} 1_B(r\xi)\rho_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

This decomposition of ρ is unique in the following sense: if (λ, ρ_ξ) and (λ', ρ'_ξ) both have the properties above, then we can find a measurable function $c(\xi)$ on S such that $0 < c(\xi) < \infty$, $\lambda'(d\xi) = c(\xi)\lambda(d\xi)$, and, for λ-almost all $\xi \in S$, $c(\xi)\rho'_\xi(dr) = \rho_\xi(dr)$.

This (λ, ρ_ξ) is called a *polar decomposition* of ρ.

Theorem 3.12. *Let $\mu \in ID(\mathbb{R}^d)$. Then $\mu \in B(\mathbb{R}^d)$ if and only if, for the triplet (A, ν, γ) of μ, ν has a polar decomposition $(\lambda, l_\xi(r)dr)$ such that $l_\xi(r)$ is measurable in ξ and completely monotone in r on $(0,\infty)$ (A and γ have no restriction). Further, $B(\mathbb{R}^d)$ equals $\mathfrak{R}(\Psi_{-1})$, the range of Ψ_{-1}.*

Here we say that a function $h(r)$ is *completely monotone* on $(0,\infty)$ if it is infinitely differentiable on $(0,\infty)$ and $(-1)^m(d/dr)^m h(r) \geq 0$ for $m = 0, 1, \ldots$.

Theorem 3.13. *Let $\mu \in ID(\mathbb{R}^d)$. Then $\mu \in T(\mathbb{R}^d)$ if and only if, for (A, ν, γ) of μ, ν has a polar decomposition $(\lambda, k_\xi(r)r^{-1}dr)$ such that $k_\xi(r)$ is measurable in ξ and completely monotone in r on $(0, \infty)$ (A and γ have no restriction). We have two expressions $T(\mathbb{R}^d) = \mathfrak{R}(\Psi_0)$ and $T(\mathbb{R}^d) = \Psi_{-1}(L(\mathbb{R}^d))$.*

For general $\alpha \in (-\infty, 2)$ we can give a description of the triplet of μ in $\mathfrak{R}(\Psi_\alpha)$, extending Theorems 3.12 and 3.13 ([49, 52]). It is shown that the class $\mathfrak{R}(\Psi_\alpha)$ is strictly decreasing in α. For $0 < \alpha < 2$, $\mathfrak{R}(\Psi_\alpha)$ is related with tempered stable distributions of index α introduced by Rosiński [38].

Let us introduce another family of functions. For $-\infty < \beta < \alpha < \infty$ let $g_{\beta,\alpha}(t) = (\Gamma(\alpha-\beta))^{-1}\int_t^1(1-u)^{\alpha-\beta-1}u^{-\alpha-1}du$ for $0 < t \le 1$ and $b_{\beta,\alpha} = g_{\beta,\alpha}(0+)$. We have $b_{\beta,\alpha} = \Gamma(-\alpha)/\Gamma(-\beta)$ for $\alpha < 0$ and $b_{\beta,\alpha} = \infty$ for $\alpha \ge 0$. Let $t = f_{\beta,\alpha}(s)$, $0 \le s < b_{\beta,\alpha}$, be the inverse function of $s = g_{\beta,\alpha}(t)$, $0 < t \le 1$. Define $f_{\beta,\alpha}(s) = 0$ for $s \notin [0, b_\alpha)$. As $s \to \infty$, $f_{\beta,0}(s) \sim e^{-\Gamma(-\beta)s}$ for $\beta < 0 = \alpha$ and $f_{\beta,\alpha}(s) \sim s^{-1/\alpha}$ for $\alpha > 0$ and $\beta < \alpha$, ignoring non-zero constant multiples. For $\beta < 1 = \alpha$ the assumption in (iii) of Theorem 3.7 is satisfied. We write $\Phi_{\beta,\alpha}$ for $\Phi_{f_{\beta,\alpha}}$. For $\beta < \alpha < 0$ we have $\mathfrak{D}^0(\Phi_{\beta,\alpha}) = ID(\mathbb{R}^d)$. For $\alpha \ge 0$ and $\beta < \alpha$, $\mathfrak{D}(\Phi_{\beta,\alpha})$ and its variants are described by Theorems 3.7 and 3.8; they are independent of β and coincide with $\mathfrak{D}(\Psi_\alpha)$ and its variants. Noting that $f_{-1,0}(s) = e^{-s}$ and writing $\Phi_{-1,0} = \Phi$, we see in Subsection 2.1 that Φ has range $\mathfrak{R}(\Phi)$ identical with the class $L(\mathbb{R}^d)$ of selfdecomposable distributions, whose triplet (A, ν, γ) is such that ν has polar decomposition given in Section 1 and A and γ have no restriction. The range $\mathfrak{R}(\Phi_{\beta,\alpha})$ depends not only on α but also on β; polar decomposition of the Lévy measure of $\mu \in \mathfrak{R}(\Phi_{\beta,\alpha})$ is expressible by the use of fractional or Riemann–Liouville integrals ([52]).

Consider mappings Φ_f and Φ_g for two locally square-integrable functions f and g on $[0, \infty)$. We define the composite mapping $\Phi_g\Phi_f$ by $\mathfrak{D}(\Phi_g\Phi_f) = \{\mu \in \mathfrak{D}(\Phi_f) \colon \Phi_f(\mu) \in \mathfrak{D}(\Phi_g)\}$ and $(\Phi_g\Phi_f)(\mu) = \Phi_g(\Phi_f(\mu))$ for $\mu \in \mathfrak{D}(\Phi_g\Phi_f)$.

Theorem 3.14. $\Psi_\alpha = \Psi_\beta\Phi_{\beta,\alpha} = \Phi_{\beta,\alpha}\Psi_\beta$ *for* $-\infty < \beta < \alpha < \infty$.

Letting $\beta = -1$ and $\alpha = 0$, we obtain $\Psi_0 = \Upsilon\Phi = \Phi\Upsilon$.

The foregoing theorems in this subsection were proved by [2, 49].

As special cases of $f_{\beta,\alpha}$, $f_{\alpha-1,\alpha}(s) = (1+\alpha s)^{-1/\alpha}$ for $\alpha \ne 0$, and $f_{\beta,-1}(s) = 1 - (\Gamma(-\beta)s)^{1/(-\beta-1)}$ for $\beta < -1$. Thus $f_{-2,-1}(s) = 1 - s$ and $b_{-2,-1} = 1$, and the following class is related. Let $U(\mathbb{R}^d)$ be the class of $\mu \in ID(\mathbb{R}^d)$ such that, for every $b \in (0,1)$, there exists $\rho \in ID(\mathbb{R}^d)$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(bz)^b\,\widehat{\rho}(z)$; $U(\mathbb{R}^d)$ is called the *Jurek class*, which is a superclass of $L(\mathbb{R}^d)$.

Theorem 3.15 (Jurek [18])**.** *The class $U(\mathbb{R}^d)$ equals $\mathfrak{R}(\Phi_{-2,-1})$. A distribution $\mu \in ID(\mathbb{R}^d)$ belongs to $U(\mathbb{R}^d)$ if and only if, in its triplet (A, ν, γ), ν has polar decomposition $(\lambda, l_\xi(r)dr)$ such that $l_\xi(r)$ is measurable in ξ and decreasing and right-continuous on $(0, \infty)$ in r (A and γ have no restriction).*

3.4. **Decreasing sequences of classes of infinitely divisible distributions.** Urbanik [56, 57] introduced a decreasing sequence beginning with the class $L(\mathbb{R}^d)$ of selfdecomposable distributions:

$$L(\mathbb{R}^d) = L_0(\mathbb{R}^d) \supset L_1(\mathbb{R}^d) \supset L_2(\mathbb{R}^d) \supset \cdots \supset L_\infty(\mathbb{R}^d)$$

and the study by Sato [40] and others followed. We define $L_0(\mathbb{R}^d) = L(\mathbb{R}^d)$ and then, for positive integers m, $L_m(\mathbb{R}^d)$ as the class of distributions μ such that, for every $b \in (0,1)$, there exists $\rho \in L_{m-1}(\mathbb{R}^d)$ satisfying (1.2). Further define $L_\infty(\mathbb{R}^d) = \bigcap_{0\le m<\infty} L_m(\mathbb{R}^d)$. These classes can be defined also from some kind of limit theorems.

Theorem 3.16. *The class $L_\infty(\mathbb{R}^d)$ is the smallest class that is closed under convolution and weak convergence and contains the class $S(\mathbb{R}^d)$ of stable distributions. A distribution $\mu \in ID(\mathbb{R}^d)$ with triplet (A,ν,γ) belongs to $L_\infty(\mathbb{R}^d)$ if and only if ν has decomposition* (1.5) *such that $h_\xi(s) = k_\xi(e^s)$ is completely monotone on $(0,\infty)$ in s (A and γ have no restriction). An alternative expression of the condition on ν is that there exist a measure Γ on the interval $(0,2)$ satisfying $\int_{(0,2)}(\alpha^{-1}+(2-\alpha)^{-1})\Gamma(d\alpha) < \infty$ and a system $\{\lambda_\alpha\colon \alpha\in(0,2)\}$ of probability measures on S measurable in α such that*

$$\nu(B) = \int_{(0,2)} \Gamma(d\alpha)\int_S \lambda_\alpha(d\xi)\int_0^\infty 1_B(r\xi)r^{-\alpha-1}dr, \quad B\in\mathcal{B}(\mathbb{R}^d).$$

We mentioned that $L(\mathbb{R}^d)$ is the range of $\Phi = \Phi_{-1,0}$. The class $L_m(\mathbb{R}^d)$ can be expressed as the range of the $(m+1)$-fold composite Φ^{m+1} of Φ and also as the range of Φ_f for some f:

Theorem 3.17. *We have $L_m(\mathbb{R}^d) = \mathfrak{R}(\Phi^{m+1})$ for $m = 0,1,2,\ldots < \infty$ and $\Phi^{m+1} = \Phi_f$ for $f(s) = \exp[-((m+1)!\,s)^{1/(m+1)}]$.*

The domain $\mathfrak{D}(\Phi_f)$ for the f above is treated in Theorem 3.8. Theorem 3.17 is essentially by Jurek [17]. We can extend the discrete parameter family $\{L_m(\mathbb{R}^d)\colon m = 0,1,\ldots\}$ to a continuous parameter family $\{L_p(\mathbb{R}^d)\colon -1<p<\infty\}$ strictly decreasing in p, letting $L_p(\mathbb{R}^d) = \mathfrak{R}(\Phi_f)$ with $f(s) = \exp[-(\Gamma(p+2)s)^{1/(p+1)}]$. This family was introduced by Nguyen Van Thu [54] by another method. Using fractional integrals, we can determine the Lévy measures of distributions in $L_p(\mathbb{R}^d)$ ([52]).

Just like $L_m(\mathbb{R}^d)$ from $L(\mathbb{R}^d)$, we can construct a sequence

$$U(\mathbb{R}^d) = U_0(\mathbb{R}^d) \supset U_1(\mathbb{R}^d) \supset U_2(\mathbb{R}^d) \supset \cdots \supset U_\infty(\mathbb{R}^d),$$

from the Jurek class $U(\mathbb{R}^d)$, using the formula $\widehat{\mu}(z) = \widehat{\mu}(bz)^b\,\widehat{\rho}(z)$ in place of $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z)$. Then it is proved that $U_m(\mathbb{R}^d) = \mathfrak{R}((\Phi_{-2,-1})^{m+1})$ for $m = 0,1,2,\ldots < \infty$.

More generally, given a locally square-integrable function f on $[0,\infty)$, we can define a decreasing sequence

$$\mathfrak{R}(\Phi_f) \supset \mathfrak{R}(\Phi_f^2) \supset \mathfrak{R}(\Phi_f^3) \supset \cdots \supset \bigcap_{0\le m<\infty} \mathfrak{R}(\Phi_f^m). \tag{3.7}$$

Let $\mathfrak{R}_\infty(\Phi_f)$ denote the final class in (3.7). Often we obtain $\mathfrak{R}_\infty(\Phi_f) = L_\infty(\mathbb{R}^d)$.

Theorem 3.18. *Let $0 < a \le \infty$ and let $p(u)$ be a positive decreasing function on $(0,a)$ satisfying $\int_0^a (1+u^2)p(u)du < \infty$. Define $g(t) = \int_t^a p(u)du$ for $0<t<a$ and $b = g(0+)$. Let $t = f(s)$, $0<s<b$, be the inverse function of $s = g(t)$, $0<t<a$. We define $f(s) = 0$ for $s\notin(0,b)$. Then $\mathfrak{D}(\Phi_f) = ID(\mathbb{R}^d)$ and $\mathfrak{R}_\infty(\Phi_f) = L_\infty(\mathbb{R}^d)$.*

The mappings Ψ_{-1} and $\Phi_{-2,-1}$ are examples of Φ_f satisfying the assumption of this theorem; the corresponding sequence (3.7) begins with $B(\mathbb{R}^d)$ and $U(\mathbb{R}^d)$, respectively. The result $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$ is an assertion by Jurek [19]. More generally, the mappings Ψ_α with $-1\le\alpha<0$ and $\Phi_{\beta,\alpha}$ with $-1\le\alpha<0$ and $\beta\le\alpha-1$ satisfy the assumption of Theorem 3.18.

If $p(u) = e^{-u^2}$, $a = \infty$, and $b = \sqrt{\pi}/2$ in Theorem 3.18, then the corresponding Φ_f is denoted by $\mathcal{G}$ and distributions in $\mathfrak{R}(\mathcal{G})$ are called *generalized type G distributions*. Symmetric distributions in $\mathfrak{R}(\mathcal{G})$ are called *type G distributions* in Maejima and Rosiński [30, 31].

We can show the following result, which is not a consequence of Theorem 3.18.

Theorem 3.19. $\mathfrak{R}_\infty(\Psi_0) = L_\infty(\mathbb{R}^d)$.

This is the final class of the sequence (3.7) beginning with $T(\mathbb{R}^d)$. Theorems 3.18 and 3.19 are shown by Maejima and Sato [33]. We note a trivial fact that if $f(s) = 1_{[0,1]}(s)$, then $\mathfrak{R}(\Phi_f^{m+1}) = ID(\mathbb{R}^d)$ for $m = 0, 1, \ldots$ and hence $\mathfrak{R}_\infty(\Phi_f) = ID(\mathbb{R}^d)$. We can also construct an example of Φ_f such that $\mathfrak{R}_\infty(\Phi_f) \subsetneqq L_\infty(\mathbb{R}^d)$.[1]

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[1](Added in translation) See K. Sato, Description of limits of ranges of iterations of stochastic integral mappings of infinitely divisible distributions, ALEA Lat. Am. J. Probab. Math. Stat. **8** (2011), 1-17.

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