Fractional integrals and extensions of selfdecomposability

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Abstract After characterizations of the class L of selfdecomposable distributions on $\mathbb{R}^d$ are recalled, the classes $K_{p,\alpha}$ and $L_{p,\alpha}$ with two continuous parameters $0 < p < \infty$ and $-\infty < \alpha < 2$ satisfying $K_{1,0} = L_{1,0} = L$ are introduced as extensions of the class L. They are defined as the classes of distributions of improper stochastic integrals $\int_0^{\infty-} f(s)dX_s^{(\rho)}$, where $f(s)$ is an appropriate non-random function and $X_s^{(\rho)}$ is a Lévy process on $\mathbb{R}^d$ with distribution ρ at time 1. The description of the classes is given by characterization of their Lévy measures, using the notion of monotonicity of order p based on fractional integrals of measures, and in some cases by addition of the condition of zero mean or some weaker conditions that are newly introduced—having weak mean 0 or having weak mean 0 absolutely. The class $L_{n,0}$ for a positive integer n is the class of n times selfdecomposable distributions. Relations among the classes are studied. The limiting classes as $p \to \infty$ are analyzed. The Thorin class T, the Goldie–Steutel–Bondesson class B, and the class L_∞ of completely selfdecomposable distributions, which is the closure (with respect to convolution and weak convergence) of the class $\mathfrak{S}$ of all stable distributions, appear in this context. Some subclasses of the class L_∞ also appear. The theory of fractional integrals of measures is built. Many open questions are mentioned.

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1 Introduction

1.1 Characterizations of selfdecomposable distributions

A distribution μ on $\mathbb{R}^d$ is called infinitely divisible if, for each positive integer n, there is a distribution μ_n such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n},$$

where $*$ denotes convolution. The class of infinitely divisible distributions on $\mathbb{R}^d$ is denoted by $ID = ID(\mathbb{R}^d)$. Let $C_\mu(z)$, $z \in \mathbb{R}^d$, denote the cumulant function of $\mu \in ID$, that is, the unique complex-valued continuous function on $\mathbb{R}^d$ with $C_\mu(0) = 0$ such that the characteristic function $\widehat{\mu}(z)$ of μ is expressed as $\widehat{\mu}(z) = e^{C_\mu(z)}$. If $\mu \in ID$, then $C_\mu(z)$ is expressed as

$$C_\mu(z) = -\frac{1}{2}\langle z, A_\mu z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\le 1\}}(x))\nu_\mu(dx) + i\langle \gamma_\mu, z\rangle. \quad (1.1)$$

Here $\langle z,x\rangle$ is the canonical inner product of z and x in $\mathbb{R}^d$, $|x| = \langle x,x\rangle^{1/2}$, $1_{\{|x|\le 1\}}$ is the indicator function of the set $\{|x| \le 1\}$, A_μ is a $d \times d$ symmetric nonnegative-definite matrix, called the Gaussian covariance matrix of μ, ν_μ is a measure on $\mathbb{R}^d$ satisfying $\nu_\mu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu_\mu(dx) < \infty$, called the Lévy measure of μ, and γ_μ is an element of $\mathbb{R}^d$. The triplet $(A_\mu, \nu_\mu, \gamma_\mu)$ is uniquely determined by μ. Conversely, to any triplet (A, ν, γ) there corresponds a unique $\mu \in ID$ such that $A = A_\mu$, $\nu = \nu_\mu$, and $\gamma = \gamma_\mu$. Throughout this article A_μ, ν_μ, and γ_μ are used in this sense.

A distribution μ on $\mathbb{R}^d$ is called *selfdecomposable* if, for any $b > 1$, there is a distribution μ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z), \qquad z \in \mathbb{R}^d. \quad (1.2)$$

Let $L = L(\mathbb{R}^d)$ denote the class of selfdecomposable distributions on $\mathbb{R}^d$. It is characterized in the following four ways.

(a) A distribution μ on $\mathbb{R}^d$ is selfdecomposable if and only if $\mu \in ID$ and its Lévy measure ν_μ has a radial (or polar) decomposition

$$\nu_\mu(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)r^{-1}k_\xi(r)dr \quad (1.3)$$

for Borel sets B in $\mathbb{R}^d$, where λ is a finite measure on the unit sphere $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ (if $d = 1$, then S is a two-point set $\{1, -1\}$) and $k_\xi(r)$ is a nonnegative function measurable in ξ and decreasing and right-continuous in r. (See Proposition 3.1 for exact formulation of radial decomposition.)

(b) Let $\{Z_k\colon k=1,2,\dots\}$ be independent random variables on $\mathbb{R}^d$ and $Y_n = \sum_{k=1}^n Z_k$. Suppose that there are $b_n > 0$ and $\gamma_n \in \mathbb{R}^d$ for $n=1,2,\dots$ such that the law of $b_nY_n+\gamma_n$ weakly converges to a distribution μ as $n\to\infty$ and that $\{b_nZ_k\colon k=1,\dots,n;\ n=1,2,\dots\}$ is a null array (that is, for any $\varepsilon>0$, $\max_{1\le k\le n} P(|b_nZ_k|>\varepsilon)\to 0$ as $n\to\infty$). Then $\mu\in L$. Conversely, any $\mu\in L$ is obtained in this way.

(c) Given $\rho\in ID$, let $\{X_t^{(\rho)}\colon t\ge 0\}$ be a Lévy process on $\mathbb{R}^d$ (that is, a stochastic process continuous in probability, starting at 0, with time-homogeneous independent increments, with cadlag paths) having distribution ρ at time 1. If $\int_{|x|>1}\log|x|\rho(dx)<\infty$, then the improper stochastic integral $\int_0^{\infty-} e^{-s}dX_s^{(\rho)}$ is definable and its distribution

$$\mu = \mathscr{L}\left(\int_0^{\infty-} e^{-s}dX_s^{(\rho)}\right) \tag{1.4}$$

is selfdecomposable. Here $\mathscr{L}(Y)$ denotes the distribution (law) of a random element Y. Conversely, any $\mu\in L$ is obtained in this way. On the other hand, if $\int_{|x|>1}\log|x|\rho(dx)=\infty$, then $\int_0^{\infty-} e^{-s}dX_s^{(\rho)}$ is not definable. (See Section 3.4 for improper stochastic integrals.)

To see that μ of (1.4) is selfdecomposable, notice that

$$\int_0^{\infty-} e^{-s}dX_s^{(\rho)} = \int_0^{\log b} e^{-s}dX_s^{(\rho)} + \int_{\log b}^{\infty-} e^{-s}dX_s^{(\rho)} = I_1+I_2,$$

I_1 and I_2 are independent, and

$$I_2 = \int_0^{\infty-} e^{-\log b-s}dX_{\log b+s}^{(\rho)} = b^{-1}\int_0^{\infty-} e^{-s}dY_s^{(\rho)},$$

where $\{Y_s^{(\rho)}\}$ is identical in law with $\{X_s^{(\rho)}\}$, and hence μ satisfies (1.2).

(d) Let $\{Y_t\colon t\ge 0\}$ be an additive process on $\mathbb{R}^d$, that is, a stochastic process continuous in probability with independent increments, with cadlag paths, and with $Y_0=0$. If, for some $H>0$, it is H-selfsimilar (that is, for any $a>0$, the two processes $\{Y_{at}\colon t\ge 0\}$ and $\{a^HY_t\colon t\ge 0\}$ have an identical law), then the distribution μ of Y_1 is in L. Conversely, for any $\mu\in L$ and $H>0$, there is a process $\{Y_t\colon t\ge 0\}$ satisfying these conditions and $\mathscr{L}(Y_1)=\mu$.

Historically, selfdecomposable distributions were introduced by Lévy [18] in 1936 and written in his 1937 book [19] under the name “lois-limites”, to characterize the limit distributions in (b). Lévy wrote in [18, 19] that this characterization problem had been posed by Khintchine, and Khintchine’s book [16] in 1938 called these distributions “of class L”. The book [9] of Gnedenko and Kolmogorov uses the same naming. Loève’s book [20] uses the name “selfdecomposable”.

The property (c) gives a characterization of the stationary distribution of an Ornstein–Uhlenbeck type process (sometimes called an Ornstein–Uhlenbeck process driven by a Lévy process) $\{V_t\colon t\ge 0\}$ defined by

$$V_t = e^{-t}V_0 + \int_0^t e^{s-t}\,dX_s^{(\rho)},$$

where V_0 and $\{X_t^{(\rho)} : t \geq 0\}$ are independent. The stationary Ornstein–Uhlenbeck type process and the selfsimilar process in the property (d) are connected via the so-called Lamperti transformation (see [11], [26]). For historical facts concerning (c) see [33], pp. 54–55.

The proofs of (a)–(d) and many examples of selfdecomposable distributions are found in Sato's book [39].

The main purpose of the present article is to give two families of subclasses of ID, with two continuous parameters, related to L, using improper stochastic integrals and extending the characterization (c) of L.

1.2 Nested classes of multiply selfdecomposable distributions

If $\mu \in L$, then, for any $b > 1$, the distribution μ_b in (1.2) is infinitely divisible and uniquely determined by μ and b. If $\mu \in L$ and $\mu_b \in L$ for all $b > 1$, then μ is called *twice selfdecomposable*. Let n be a positive integer ≥ 3. A distribution μ is called *n times selfdecomposable*, if $\mu \in L$ and if μ_b is $n-1$ times selfdecomposable. Let $L_{1,0} = L_{1,0}(\mathbb{R}^d) = L(\mathbb{R}^d)$ and let $L_{n,0} = L_{n,0}(\mathbb{R}^d)$ be the class of n times selfdecomposable distributions on $\mathbb{R}^d$. Then we have

$$ID \supset L = L_{1,0} \supset L_{2,0} \supset L_{3,0} \supset \cdots . \tag{1.5}$$

These classes and the class $L_\infty(\mathbb{R}^d)$ in Section 1.4 were introduced by Urbanik [52, 53] and studied by Sato [37] and others. (In [37, 52, 53] the class $L_{n,0}$ is written as L_{n-1}, but this notation is inconvenient in this article.)

An n times selfdecomposable distribution is characterized by the property that $\mu \in ID$ with Lévy measure ν_μ having radial decomposition (1.3) in (a) with $k_\xi(r) = h_\xi(\log r)$ for some function $h_\xi(y)$ monotone of order n for each ξ (see Section 1.5 and Proposition 2.11 for the monotonicity of order n). In the propeerty (b), $\mu \in L_{n,0}$ is characterized by the property that $\mathscr{L}(Z_k) \in L_{n-1,0}$ for $k = 1, 2, \ldots$. In (c), $\mu \in L_{n,0}$ is characterized by $\rho \in L_{n-1,0}$ in (1.4). A direct generalization of (1.4) using $\exp(-s^{1/n})$ or, equivalently, $\exp(-(n!\,s)^{1/n})$ in place of e^{-s} is also possible. In (d), $\mu \in L_{n,0}$ if and only if, for any H, the corresponding process $\{Y_t : t \geq 0\}$ satisfies $\mathscr{L}(Y_t - Y_s) \in L_{n-1,0}$ for $0 < s < t$. The proofs are given in [12, 25, 33, 37].

1.3 Continuous-parameter extension of multiple selfdecomposability

In 1980s Nguyen Van Thu [49, 50, 51] defined a continuous-parameter extension of $L_{n,0}$, replacing the positive integer n by a real number $p > 0$. He introduced fractional times multiple selfdecomposability and used fractional integrals and fractional difference quotients. On one hand he extended the definition of n times selfdecomposability based on (1.2) to fractional times selfdecomposability in the form of infinite products. On the other hand he extended essentially the formula (1.4) in the characterization (c), considering its Lévy measure.

Directly using improper stochastic integrals with respect to Lévy processes, we will define and study the decreasing classes $L_{p,0}$ for $p > 0$, which generalize the nested classes $L_{n,0}$ for $n = 1, 2, \ldots$. Thus the results of Thu will be reformulated as a special case in a family $L_{p,\alpha}$ with two continuous parameters $0 < p < \infty$ and $-\infty < \alpha < 2$. The definition of $L_{p,\alpha}$ will be given in Section 1.6.

1.4 Stable distributions and class L_∞

Let μ be a distribution on $\mathbb{R}^d$. Let $0 < \alpha \le 2$. We say that μ is *strictly α-stable* if $\mu \in ID$ and, for any $t > 0$, $\widehat{\mu}(z)^t = \widehat{\mu}(t^{1/\alpha}z)$, $z \in \mathbb{R}^d$. We say that μ is *α-stable* if $\mu \in ID$ and, for any $t > 0$, there is $\gamma_t \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^t = \widehat{\mu}(t^{1/\alpha}z)\exp(i\langle\gamma_t, z\rangle)$, $z \in \mathbb{R}^d$. (When μ is a δ-distribution, this terminology is not the same as in Sato [39].) Let $\mathfrak{S}_\alpha^0 = \mathfrak{S}_\alpha^0(\mathbb{R}^d)$ and $\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbb{R}^d)$ be the class of strictly α-stable distributions on $\mathbb{R}^d$ and the class of α-stable distributions on $\mathbb{R}^d$, respectively. Let $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^d)$ be the class of stable distributions on $\mathbb{R}^d$. That is, $\mathfrak{S} = \bigcup_{0<\alpha\le 2}\mathfrak{S}_\alpha$. A distribution $\mu \in ID$ is in $\mathfrak{S}_2$ if and only if $\nu_\mu = 0$, that is, μ is Gaussian. A distribution $\mu \in ID$ is in $\mathfrak{S}_\alpha$ with $0 < \alpha < 2$ if and only if $A_\mu = 0$ and ν_μ has a radial decomposition (1.3) with $k_\xi(r) = r^{-\alpha}$. A distribution $\mu \in \mathfrak{S}_\alpha$ with $1 < \alpha \le 2$ is in $\mathfrak{S}_\alpha^0$ if and only if μ has mean 0. A distribution $\mu \in \mathfrak{S}_1$ is in $\mathfrak{S}_1^0$ if and only if ν_μ has a radial decomposition (1.3) with $k_\xi(r) = r^{-1}$ and $\int_S \xi\lambda(d\xi) = 0$. A distribution $\mu \in \mathfrak{S}_\alpha$ with $0 < \alpha < 1$ is in $\mathfrak{S}_\alpha^0$ if and only if it is driftless in the sense that

$$C_\mu(z) = \int_S \lambda(d\xi)\int_0^\infty (e^{i\langle r\xi, z\rangle} - 1)r^{-\alpha-1}dr, \qquad z \in \mathbb{R}^d.$$

Lots of results are accumulated on stable distributions and processes. To mention one of them, the asymptotic behavior of the density of $\mu \in \mathfrak{S}_\alpha(\mathbb{R}^d)$, $d \ge 2$, $\alpha \in (0, 2)$, sensitively depends on the radial direction and exhibits amazing diversity, as Watanabe [54] shows.

Let $L_\infty = L_\infty(\mathbb{R}^d)$ denote the smallest class that is closed under convolution and weak convergence and contains $\mathfrak{S}(\mathbb{R}^d)$. This class was introduced by Urbanik [52, 53] and reformulated by Sato [37]. If $\mu \in L_\infty$, then $\mu \in ID$ with Lévy measure ν_μ being such that

$$\nu_\mu(B) = \int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr \tag{1.6}$$

for Borel sets B in $\mathbb{R}^d$, where Γ is a measure on the open interval $(0,2)$ satisfying

$$\int_{(0,2)} (\alpha^{-1} + (2-\alpha)^{-1}) \Gamma(d\alpha) < \infty \tag{1.7}$$

and $\{\lambda_\alpha \colon \alpha \in (0,2)\}$ is a measurable family of probability measures on S. This Γ is determined by ν_μ, and λ_α is determined by ν_μ up to α of Γ-measure 0. Conversely, if a measure ν on $\mathbb{R}^d$ is expressed by the right-hand side of (1.6) with some Γ and λ_α satisfying the conditions above, then, for any A and γ, (A, ν, γ) is the triplet of some $\mu \in L_\infty$.

We will also use the class $L_\infty^E = L_\infty^E(\mathbb{R}^d)$ for a Borel subset E of the open interval $(0,2)$; this is the class of $\mu \in L_\infty$ whose measure Γ is concentrated on E.

Another characterization of $L_\infty(\mathbb{R}^d)$ is that $\mu \in L_\infty$ if and only if $\mu \in L$ and ν_μ has a radial decomposition (1.3) with $k_\xi(r) = h_\xi(\log r)$ where h_ξ is a completely monotone function on $\mathbb{R}$ for each ξ. Hence we have

$$L_\infty = \bigcap_{n=1,2,\ldots} L_{n,0}. \tag{1.8}$$

Thus distributions in L_∞ are sometimes called *completely selfdecomposable*.

Zinger [57] introduced a subclass $\mathscr{P}_r$ (r being a positive integer) of the class $L(\mathbb{R})$; it is defined to be the class of limit distributions μ in (b) of Section 1.1 such that $\{\mathscr{L}(Z_k) \colon k = 1,2,\ldots\}$ consists of at most r different distributions on $\mathbb{R}$. It is known that $\mathscr{P}_1 = \mathfrak{S}(\mathbb{R})$ and that $\mu \in \mathscr{P}_2$ if and only if μ is the convolution of at most two stable distributions. In [57] a beautiful explicit description of the Lévy measures of distributions in $\mathscr{P}_r$ is given and it is shown that a distribution in $\mathscr{P}_r$ with $r \geq 3$ is not necessarily the convolution of stable distributions on $\mathbb{R}$. Any μ in $\mathscr{P}_r$ is the convolution of at most r semi-stable distributions of a special form. However, no other characterization of $\mathscr{P}_r$ exists, as far as the author knows.

1.5 Fractional integrals

The key concept to connect the representation of Lévy measures for the class $L(\mathbb{R}^d)$ and that for the class $L_\infty(\mathbb{R}^d)$ is monotonicity of order $p \in (0,\infty)$. It is defined by using the notion of fractional integrals or Riemann–Liouville integrals. Let us write

$$\Gamma_p = \Gamma(p), \quad c_p = 1/\Gamma(p)$$

throughout this article. The fractional integral of order $p > 0$ of a function $f(s)$ on $\mathbb{R}$ in a suitable class is given by

$$c_p \int_r^\infty (s-r)^{p-1} f(s)ds,$$

which is the interpolation ($1 \le p < \infty$) and extrapolation ($0 < p \le 1$) of the n times integration

$$\int_r^\infty ds_n \int_{s_n}^\infty ds_{n-1} \cdots \int_{s_2}^\infty f(s_1)ds_1 = \frac{1}{(n-1)!} \int_r^\infty (s-r)^{n-1} f(s)ds.$$

However, we need to use fractional integrals of measures. Our definition is as follows.

Let

$$\mathbb{R}_+ = [0,\infty), \quad \mathbb{R}_+^\circ = (0,\infty)$$

and $\mathscr{B}(E)$ for the class of Borel sets in a space E. A measure σ is said to be locally finite on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] if $\sigma([a,b]) < \infty$ for all a, b with $-\infty < a < b < \infty$ [resp. $0 < a < b < \infty$]. Let $p > 0$. For a measure σ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$], let

$$\widetilde{\sigma}(E) = c_p \int_E dr \int_{(r,\infty)} (s-r)^{p-1} \sigma(ds), \quad E \in \mathscr{B}(\mathbb{R}) \text{ [resp. } \mathscr{B}(\mathbb{R}_+^\circ)]. \tag{1.9}$$

Let $\mathfrak{D}(I^p)$ [resp. $\mathfrak{D}(I_+^p)$] be the class of locally finite measures σ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] such that $\widetilde{\sigma}$ is a locally finite measure on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Define

$$I^p \sigma(E) = \widetilde{\sigma}(E), \quad E \in \mathscr{B}(\mathbb{R}) \quad \text{[resp. } I_+^p \sigma(E) = \widetilde{\sigma}(E), \quad E \in \mathscr{B}(\mathbb{R}_+^\circ)]$$

for $\sigma \in \mathfrak{D}(I^p)$ [resp. $\mathfrak{D}(I_+^p)$]. Thus I^p and I_+^p are mappings from measures to measures on $\mathbb{R}$ and $\mathbb{R}_+^\circ$, respectively. $\mathfrak{D}(I^p)$ and $\mathfrak{D}(I_+^p)$ are their domains.

We call a $[0,\infty]$-valued function $f(r)$ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] *monotone of order p* on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] if

$$f(r) = c_p \int_{(r,\infty)} (s-r)^{p-1} \sigma(ds) \tag{1.10}$$

with some $\sigma \in \mathfrak{D}(I^p)$ [resp. $\mathfrak{D}(I_+^p)$]. As will be shown in Example 2.17, functions monotone of order $p \in (0,1)$ have, in general, quite different properties from functions monotone of order $p \in [1,\infty)$. We call $f(r)$ *completely monotone* on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] if it is monotone of order p on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] for all $p > 0$. This definition of complete monotonicity differs from the usual one in that positive constant functions are not completely monotone. Typical completely monotone functions on $\mathbb{R}$ and $\mathbb{R}_+^\circ$ are e^{-r} and $r^{-\alpha}$ ($\alpha > 0$), respectively.

The properties of fractional integrals of functions are studied in M. Riesz [32], Ross (ed.) [35], Samko, Kilbas, and Marichev [36], Kamimura [15], and others. Williamson [56] studied fractional integrals of measures on $\mathbb{R}_+^\circ$ for $p \ge 1$ and introduced the concept of p-times monotonicity. But we do not assume any knowledge of them.

In Sections 2.1–2.3 we build the theory of the fractional integral mappings I^p and I_+^p for $p \in (0,\infty)$ from the point of view that they are mappings from measures to measures. A basic relation is the semigroup property $I^q I^p = I^{p+q}$ and $I_+^q I_+^p = I_+^{p+q}$.

An important property that both I^p and I^p_+ are one-to-one is proved. The relation between the theories on $\mathbb{R}$ and $\mathbb{R}^\circ_+$ is not extension and restriction. We need both theories, as will be mentioned at the end of Section 6.2.

1.6 Classes $K_{p,\alpha}$ and $L_{p,\alpha}$ generated by stochastic integral mappings

The formula (1.4) gives a mapping Φ from $\rho \in ID(\mathbb{R}^d)$ to $\mu \in ID(\mathbb{R}^d)$. Thus

$$\Phi\rho = \mathscr{L}\left(\int_0^{\infty-} e^{-s} dX_s^{(\rho)}\right). \tag{1.11}$$

The domain of Φ is the class of ρ for which the improper stochastic integral in (1.11) is definable.

For functions $f(s)$ in a suitable class, we are interested in the mapping Φ_f from $\rho \in ID$ to $\mu \in ID$ defined by

$$\mu = \Phi_f \rho = \mathscr{L}\left(\int_0^{\infty-} f(s) dX_s^{(\rho)}\right). \tag{1.12}$$

The domain $\mathfrak{D}(\Phi_f)$ is the class of ρ for which the improper stochastic integral in (1.12) is definable. The range is defined by $\mathfrak{R}(\Phi_f) = \{\Phi_f\rho \colon \rho \in \mathfrak{D}(\Phi_f)\}$.

Let us consider three families of functions. For $0 < p < \infty$ and $-\infty < \alpha < \infty$ let

$$\bar{g}_{p,\alpha}(t) = c_p \int_t^1 (1-u)^{p-1} u^{-\alpha-1} du, \quad 0 < t \le 1, \tag{1.13}$$

$$j_{p,\alpha}(t) = c_p \int_t^1 (-\log u)^{p-1} u^{-\alpha-1} du, \quad 0 < t \le 1, \tag{1.14}$$

$$g_\alpha(t) = \int_t^\infty u^{-\alpha-1} e^{-u} du, \quad 0 < t < \infty, \tag{1.15}$$

and $\bar{a}_{p,\alpha} = \bar{g}_{p,\alpha}(0+)$, $b_{p,\alpha} = j_{p,\alpha}(0+)$, $a_\alpha = g_\alpha(0+)$. If $\alpha < 0$, then $\bar{a}_{p,\alpha} = \Gamma_{-\alpha}/\Gamma_{p-\alpha}$, $b_{p,\alpha} = (-\alpha)^{-p}$, and $a_\alpha = \Gamma_{-\alpha}$. If $\alpha \ge 0$, then $\bar{a}_{p,\alpha} = b_{p,\alpha} = a_\alpha = \infty$. Let $t = \bar{f}_{p,\alpha}(s)$, $l_{p,\alpha}(s)$, and $f_\alpha(s)$ be the inverse functions of $s = \bar{g}_{p,\alpha}(t)$, $j_{p,\alpha}(t)$, and $g_\alpha(t)$, respectively. When $\alpha < 0$, extend $\bar{f}_{p,\alpha}(s)$ for $s \ge \bar{a}_{p,\alpha}$, $l_{p,\alpha}(s)$ for $s \ge b_{p,\alpha}$, and $f_\alpha(s)$ for $s \ge a_\alpha$ to be zero. Define

$$\bar{\Phi}_{p,\alpha} = \Phi_{\bar{f}_{p,\alpha}}, \quad \Lambda_{p,\alpha} = \Phi_{l_{p,\alpha}}, \quad \Psi_\alpha = \Phi_{f_\alpha}.$$

Sato [42] studied the mapping Ψ_α and the mapping $\Phi_{\beta,\alpha} = \Phi_{f_{\beta,\alpha}}$, $-\infty < \beta < \alpha < \infty$, for the inverse function $f_{\beta,\alpha}(s)$ of the function $g_{\beta,\alpha}(t)$ defined by

$$g_{\beta,\alpha}(t) = c_{\alpha-\beta} \int_t^1 (1-u)^{\alpha-\beta-1} u^{-\alpha-1} du, \quad 0 < t \le 1.$$

To make parametrization more convenient, we use $\bar{\Phi}_{p,\alpha} = \Phi_{\alpha-p,\alpha}$. For $\bar{\Phi}_{p,\alpha}$, $\Lambda_{p,\alpha}$, and Ψ_α, the domains will be characterized. In the analysis of the domains, asymptotic behaviors of $\bar{f}_{p,\alpha}(s)$, $l_{p,\alpha}(s)$, and $f_\alpha(s)$ for $s \to \infty$ are essential. The behaviors of $\bar{f}_{p,\alpha}(s)$ and $f_\alpha(s)$ are similar, but the behavior of $l_{p,\alpha}(s)$ is different from them. If $\alpha \geq 2$, then $\mathfrak{D}(\bar{\Phi}_{p,\alpha}) = \mathfrak{D}(\Lambda_{p,\alpha}) = \mathfrak{D}(\Psi_\alpha) = \{\delta_0\}$. So we will only consider $-\infty < \alpha < 2$. Define

$$K_{p,\alpha} = K_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}(\bar{\Phi}_{p,\alpha}), \tag{1.16}$$

$$L_{p,\alpha} = L_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}(\Lambda_{p,\alpha}). \tag{1.17}$$

It is clear that $\bar{g}_{1,\alpha}(t) = j_{1,\alpha}(t)$, and hence

$$\bar{\Phi}_{1,\alpha} = \Lambda_{1,\alpha}, \quad K_{1,\alpha} = L_{1,\alpha} \quad \text{for } -\infty < \alpha < 2. \tag{1.18}$$

Since $\bar{g}_{1,0}(t) = j_{1,0}(t) = -\log t$, $0 < t \leq 1$, and $\bar{f}_{1,0}(s) = l_{1,0}(s) = e^{-s}$, $s \geq 0$, we have

$$\bar{\Phi}_{1,0} = \Lambda_{1,0} = \Phi, \quad K_{1,0} = L_{1,0} = L \tag{1.19}$$

So $K_{p,\alpha}$ and $L_{p,\alpha}$ give extensions, with two continuous parameters, of the class L of selfdecomposable distributions. Since $l_{p,0}(s) = \exp(-(\Gamma_{p+1}s)^{1/p})$, $s \geq 0$, the class $L_{p,0}$ coincides with the class of n times selfdecomposable distributions if p is an integer n.

The following are some of the new results in this article. For any α and p with $-\infty < \alpha < 2$ and $p > 0$, any $\mu \in K_{p,\alpha}$ has Lévy measure ν_μ having a radial decomposition

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} k_\xi(r) dr \tag{1.20}$$

with $k_\xi(r)$ measurable in (ξ, r) and monotone of order p on $\mathbb{R}_+^\circ$ in r, and any $\mu \in L_{p,\alpha}$ has Lévy measure ν_μ having a radial decomposition

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} h_\xi(\log r) dr \tag{1.21}$$

with $h_\xi(y)$ measurable in (ξ, y) and monotone of order p on $\mathbb{R}$ in y. If $-\infty < \alpha < 1$, then this property of ν_μ characterizes $K_{p,\alpha}$ and $L_{p,\alpha}$. If $1 < \alpha < 2$, then this property of ν_μ combined with the property of mean 0 (that is, $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$ and $\int_{\mathbb{R}^d} x\mu(dx) = 0$) characterizes $K_{p,\alpha}$ and $L_{p,\alpha}$. We will introduce the notion of weak mean of infinitely divisible distributions in Section 3.3. If $\alpha = 1$, then the property above of ν_μ and the property of weak mean 0 characterize $K_{p,1}$; the case of $L_{p,1}$ is still open. For each fixed α, the classes $K_{p,\alpha}$ and $L_{p,\alpha}$ are strictly decreasing as p increases and at the limit there appear connections with $\mathfrak{R}(\Psi_\alpha)$ and with the class L_∞ of completely selfdecomposable distributions. Namely, define

$$K_{\infty,\alpha} = \bigcap_{0<p<\infty} K_{p,\alpha}, \quad L_{\infty,\alpha} = \bigcap_{0<p<\infty} L_{p,\alpha}. \tag{1.22}$$

It will be proved that

$$K_{\infty,\alpha} = \mathfrak{R}(\Psi_\alpha) \quad \text{for } -\infty < \alpha < 2, \tag{1.23}$$

$$L_{\infty,\alpha} = L_\infty \quad \text{for } -\infty < \alpha \le 0, \tag{1.24}$$

$$L_{\infty,\alpha} = L_\infty^E \quad \text{with } E = (\alpha, 2) \text{ for } 0 < \alpha < 1, \tag{1.25}$$

$$L_{\infty,\alpha} = L_\infty^E \cap \{\mu \colon \textstyle\int_{\mathbb{R}^d} x\mu(dx) = 0\} \quad \text{with } E = (\alpha, 2) \text{ for } 1 < \alpha < 2. \tag{1.26}$$

The case of $L_{\infty,1}$ is open.

Combined with the results in [42], the following will be shown. For any α with $-\infty < \alpha < 2$, any $\mu \in \mathfrak{R}(\Psi_\alpha)$ has Lévy measure ν_μ satisfying (1.20) in which $k_\xi(r)$ is measurable in (ξ, r) and completely monotone on $\mathbb{R}_+^\circ$ in r. If $-\infty < \alpha < 1$, then this property of ν_μ characterizes $\mathfrak{R}(\Psi_\alpha)$. If $1 < \alpha < 2$, then this property of ν_μ and the property of mean 0 characterize $\mathfrak{R}(\Psi_\alpha)$. If $\alpha = 1$, then this property of ν_μ and the property of weak mean 0 characterize $\mathfrak{R}(\Psi_1)$.

We will further establish relations among the classes and among stochastic integral mappings. The transformations of Lévy measures corresponding to Φ_f, denoted by Φ_f^L, will be examined, which gives the basis of the analysis of the ranges.

Along with the usual improper stochastic integrals Φ_f, we will use absolutely definable improper stochastic integrals and essentially definable improper stochastic integrals introduced in [41, 42, 43] (see Section 3.4). The domain $\mathfrak{D}^0(\Phi_f)$ of the former is a subclass of $\mathfrak{D}(\Phi_f)$ and the domain $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$ of the latter is a superclass of $\mathfrak{D}(\Phi_f)$. Corresponding to them the absolute range $\mathfrak{R}^0(\Phi_f)$ and the essential range $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$ are introduced. For $f = \overline{f}_{p,\alpha}$ and $f = l_{p,\alpha}$ they define $K_{p,\alpha}^0$, $K_{p,\alpha}^{\mathrm{e}}$, $L_{p,\alpha}^0$, and $L_{p,\alpha}^{\mathrm{e}}$. These classes not only help to study the classes $K_{p,\alpha}$ and $L_{p,\alpha}$, but also are interesting classes themselves.

Rosiński's study [34] of tempered stable processes concerns the Lévy processes associated with distributions in $\mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$, $0 < \alpha < 2$, with Gaussian part zero.

1.7 Remarkable subclasses of ID

We have already mentioned the subclasses L, $L_{n,0}$, $\mathfrak{S}$, L_∞, $K_{p,\alpha}$, and $L_{p,\alpha}$ of $ID(\mathbb{R}^d)$. Let us give the definitions of T, B, and U.

Let us call Vx an elementary Γ-variable [resp. elementary mixed-exponential variable, elementary compound Poisson variable] on $\mathbb{R}^d$ if x is a non-random, non-zero element of $\mathbb{R}^d$ and V is a real random variable having Γ-distribution [resp. a mixture of a finite number of exponential distributions, compound Poisson distribution whose jump size distribution is uniform on the interval $[0,a]$ for some $a > 0$]. Let $T = T(\mathbb{R}^d)$ [resp. $B = B(\mathbb{R}^d)$, $U = U(\mathbb{R}^d)$] be the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and weak convergence and containing the distributions of all elementary Γ-variables [resp. elementary mixed-exponential variables, elementary compound Poisson variables] on $\mathbb{R}^d$. We call T the *Thorin class*, B the *Goldie-Steutel-Bondesson class*, and U the *Jurek class*. It is known that

$$T = \mathfrak{R}(\Psi_0), \tag{1.27}$$
$$B = \mathfrak{R}(\Psi_{-1}), \tag{1.28}$$
$$U = \mathfrak{R}(\bar{\Phi}_{1,-1}) = K_{1,-1}. \tag{1.29}$$

See [1, 3, 13]. Concerning B and U, notice that $f_{-1}(s) = -\log s$, $0 < s \le 1$, so that

$$\Psi_{-1}\rho = \Upsilon\rho = \mathscr{L}\left(\int_0^1 (-\log s)\, dX_s^{(\rho)}\right),$$

where Υ is the mapping introduced by Barndorff-Nielsen and Thorbjørnsen [3], and that $\bar{f}_{1,-1}(s) = 1-s$, $0 \le s \le 1$, so that

$$\bar{\Phi}_{1,-1}\rho = \mathscr{L}\left(\int_0^1 (1-s)\, dX_s^{(\rho)}\right) = \mathscr{L}\left(\int_0^1 s\, dX_s^{(\rho)}\right),$$

which is the mapping of Jurek [13]. Noting (1.23), we see that

$$T = K_{\infty,0}, \tag{1.30}$$
$$B = K_{\infty,-1}. \tag{1.31}$$

Historically, the class of $\mu \in T(\mathbb{R})$ on the positive axis was introduced by Thorin [47, 48] in the naming of generalized Γ-convolutions (GGC), to show that Pareto and log-normal distributions are infinitely divisible. The class of $\mu \in B(\mathbb{R})$ on the positive axis was introduced by Bondesson [4] in the naming of generalized convolutions of mixtures of exponential distributions (g.c.m.e.d), after Goldie showed the infinite divisibility of mixtures of exponential distributions and Steutel found the description of their Lévy measures. The present formulation of $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$ is by Barndorff-Nielsen, Maejima, and Sato [1]. The class U was introduced by Jurek [13] as the class of s-selfdecomposable distributions. Our formulation of $U(\mathbb{R}^d)$ is new; we can prove its equivalence to the definition of Jurek similarly to the proof of Theorem F of [1].

See Bodesson [5] and Steutel and van Harn [46] for examples and related classes. Especially, many examples in $T(\mathbb{R})$ are known. To mention one of them, the distribution of Lévy's stochastic area of the two-dimensional Brownian motion has density $1/(\pi\cosh x)$ and belongs to $T(\mathbb{R})$ with Lévy measure $dx/(2\,|x\sinh x|)$.

2 Fractional integrals and monotonicity of order $p > 0$

2.1 Basic properties

For $\alpha \in \mathbb{R}$, let $\mathfrak{M}_\infty^\alpha(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^\alpha(\mathbb{R}_+^\circ)$] be the class of locally finite measures σ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] such that $\int_{(1,\infty)} r^\alpha \sigma(dr) < \infty$. For $\beta \in \mathbb{R}$, let $\mathfrak{M}_0^\beta(\mathbb{R}_+^\circ)$ be the class of locally finite measures σ on $\mathbb{R}_+^\circ$ such that $\int_{(0,1]} r^\beta \sigma(dr) < \infty$. Let $\mathfrak{M}^L = \mathfrak{M}^L(\mathbb{R}^d)$ be the class of measures ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. That is, $\mathfrak{M}^L(\mathbb{R}^d)$ is the class of Lévy measures of infinitely divisible distributions on $\mathbb{R}^d$. The words *increase* and *decrease* are used in the non-strict sense.

In Section 1.5 we defined the mappings I^p and I_+^p for $p > 0$ and the notion of monotonicity of order p. Let us begin with the following remarks. (i) If f is monotone of order $p > 0$ on $\mathbb{R}$, then the restriction of f to $\mathbb{R}_+^\circ$ is monotone of order p on $\mathbb{R}_+^\circ$. (ii) If f is monotone of order $p \geq 1$, then f is finite-valued and decreasing. For $p = 1$ this is obvious. For $p > 1$ this follows from Corollary 2.6 to be given later. (iii) If f is monotone of order $p \in (0,1)$, then f is finite almost everywhere, but f possibly takes the infinite value at some point and f is not necessarily decreasing. See Example 2.17 (a), (b), and (d).

Proposition 2.1. *Let $p > 0$. It holds that*

$$\mathfrak{D}(I^p) = \mathfrak{M}_\infty^{p-1}(\mathbb{R}), \tag{2.1}$$

$$\mathfrak{D}(I_+^p) = \mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ). \tag{2.2}$$

Proof. Let σ be a locally finite measure on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Let $-\infty < a < b < \infty$ [resp. $0 < a < b < \infty$]. Then $\widetilde{\sigma}$ of (1.9) satisfies

$$\begin{aligned}
\widetilde{\sigma}([a,b]) &= c_p \int_a^b dr \int_{(r,\infty)} (s-r)^{p-1} \sigma(ds) \\
&= c_p \int_{(a,\infty)} \sigma(ds) \int_a^{b\wedge s} (s-r)^{p-1} dr \\
&= c_{p+1} \int_{(b,\infty)} ((s-a)^p - (s-b)^p)\sigma(ds) + c_{p+1} \int_{(a,b]} (s-a)^p \sigma(ds),
\end{aligned}$$

which is finite if and only if $\int_{(1,\infty)} s^{p-1}\sigma(ds) < \infty$, since

$$(s-a)^p - (s-b)^p = s^p((1-a/s)^p - (1-b/s)^p) \sim p(b-a)s^{p-1}$$

as $s \to \infty$. $\square$

Corollary 2.2. *If $0 < q < p$, then $\mathfrak{D}(I^p) \subset \mathfrak{D}(I^q)$ and $\mathfrak{D}(I_+^p) \subset \mathfrak{D}(I_+^q)$.*

Proposition 2.3. *Let $p > 0$. Let $\alpha > -1$ and $\beta > 0$.*

(i) *Let $\sigma \in \mathfrak{D}(I^p)$ [resp. $\mathfrak{D}(I_+^p)$]. Then $I^p\sigma \in \mathfrak{M}_\infty^\alpha(\mathbb{R})$ [resp. $I_+^p\sigma \in \mathfrak{M}_\infty^\alpha(\mathbb{R}_+^\circ)$] if and only if $\sigma \in \mathfrak{M}_\infty^{p+\alpha}(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^{p+\alpha}(\mathbb{R}_+^\circ)$].*

(ii) *Let $\sigma \in \mathfrak{D}(I_+^p)$. Then $I_+^p\sigma \in \mathfrak{M}_0^\alpha(\mathbb{R}_+^\circ)$ if and only if $\sigma \in \mathfrak{M}_0^{p+\alpha}(\mathbb{R}_+^\circ)$.*

(iii) *Let $\sigma \in \mathfrak{D}(I^p)$. Then $\int_{(-\infty,0)} e^{\beta r}(I^p\sigma)(dr) < \infty$ if and only if $\int_{(-\infty,0)} e^{\beta s}\sigma(ds) < \infty$.*

Assertion (i) is the right-tail fattening property of I^p [resp. tail fattening property of I_+^p]. Assertion (ii) is the head thinning property of I_+^p.

Proof. (i) Let $\widetilde{\sigma} = I^p\sigma$ [resp. $I^p_+\sigma$]. We have

$$\begin{aligned}\int_1^\infty r^\alpha \widetilde{\sigma}(dr) &= c_p \int_1^\infty r^\alpha dr \int_{(r,\infty)} (s-r)^{p-1}\sigma(ds)\\ &= c_p \int_{(1,\infty)} \sigma(ds) \int_1^s r^\alpha (s-r)^{p-1} dr\\ &= c_p \int_{(1,\infty)} s^{p+\alpha}\sigma(ds) \int_{1/s}^1 u^\alpha (1-u)^{p-1} du.\end{aligned}$$

Hence $\int_{(1,\infty)} r^\alpha \widetilde{\sigma}(dr) < \infty$ if and only if $\int_{(1,\infty)} s^{p+\alpha}\sigma(ds) < \infty$.

(ii) We have

$$\begin{aligned}\int_0^1 r^\alpha (I^p_+\sigma)(dr) &= c_p \int_0^1 r^\alpha dr \int_{(r,\infty)} (s-r)^{p-1}\sigma(ds)\\ &= c_p \int_{(0,\infty)} \sigma(ds) \int_0^{1\wedge s} r^\alpha (s-r)^{p-1} dr\\ &= c_p \int_{(0,1]} f(s)\sigma(ds) + c_p \int_{(1,\infty)} g(s)\sigma(ds)\end{aligned}$$

where

$$\begin{aligned}f(s) &= \int_0^s r^\alpha (s-r)^{p-1} dr \quad \text{for } 0 < s \le 1,\\ g(s) &= \int_0^1 r^\alpha (s-r)^{p-1} dr \quad \text{for } s > 1.\end{aligned}$$

Since

$$f(s) = s^{\alpha+p} \int_0^1 u^\alpha (1-u)^{p-1} du$$

and

$$g(s) = s^{\alpha+p} \int_0^{1/s} u^\alpha (1-u)^{p-1} du \sim (\alpha+1)^{-1} s^{p-1}, \quad s \to \infty,$$

and since $\int_{(1,\infty)} s^{p-1}\sigma(ds) < \infty$, we obtain the assertion.

(iii) We have

$$\begin{aligned}\int_{-\infty}^0 e^{\beta r} (I^p\sigma)(dr) &= c_p \int_{-\infty}^0 e^{\beta r} dr \int_{(r,\infty)} (s-r)^{p-1}\sigma(ds)\\ &= c_p \int_{\mathbb{R}} \sigma(ds) \int_{-\infty}^{s\wedge 0} e^{\beta r} (s-r)^{p-1} dr\\ &= c_p \int_{(-\infty,0]} f(s)\sigma(ds) + c_p \int_{(0,\infty)} g(s)\sigma(ds),\end{aligned}$$

where

$$f(s) = \int_{-\infty}^s e^{\beta r} (s-r)^{p-1} dr \quad \text{for } s \le 0,$$

$$g(s) = \int_{-\infty}^{0} e^{\beta r}(s-r)^{p-1}dr \quad \text{for } s > 0.$$

Notice that

$$f(s) = e^{\beta s}\int_0^\infty e^{-\beta u}u^{p-1}du$$

and

$$g(s) = e^{\beta s}\int_s^\infty e^{-\beta u}u^{p-1}du \sim \beta^{-1}s^{p-1}, \quad s \to \infty.$$

Using $\int_{(1,\infty)} s^{p-1}\sigma(ds) < \infty$, we can show the result. □

Proposition 2.4. *For any* $p > 0$ *and* $q > 0$*,*

$$I^q I^p = I^{p+q} \quad \textit{and} \quad I_+^q I_+^p = I_+^{p+q}. \tag{2.3}$$

As always an equality of mappings includes the assertion that the domains of both hands are equal.

Lemma 2.5. *Let* $p > 0$ *and* $q > 0$*. If* $\sigma \in \mathfrak{D}(I^p)$ *[resp.* $\mathfrak{D}(I_+^p)$*] and* $\widetilde{\sigma} = I^p\sigma$ *[resp.* $I_+^p\sigma$*], then*

$$c_q\int_{(u,\infty)}(r-u)^{q-1}\widetilde{\sigma}(dr) = c_{p+q}\int_{(u,\infty)}(s-u)^{p+q-1}\sigma(ds) \tag{2.4}$$

for $u \in \mathbb{R}$ *[resp.* $\mathbb{R}_+^\circ$*].*

Proof. We have

$$\begin{aligned}
\int_{(u,\infty)} c_q(r-u)^{q-1}\widetilde{\sigma}(dr) &= \int_u^\infty c_q(r-u)^{q-1}dr\int_{(r,\infty)} c_p(s-r)^{p-1}\sigma(ds)\\
&= c_pc_q\int_{(u,\infty)}\sigma(ds)\int_u^s (r-u)^{q-1}(s-r)^{p-1}dr\\
&= c_pc_q\int_{(u,\infty)}(s-u)^{p+q-1}\sigma(ds)\int_0^1(1-v)^{q-1}v^{p-1}dv\\
&\qquad\text{(by change of variables } v=(s-r)/(s-u))\\
&= c_{p+q}\int_{(u,\infty)}(s-u)^{p+q-1}\sigma(ds),
\end{aligned}$$

which is (2.4). □

Proof of Proposition 2.4. We prove the first equation in (2.3), but the proof of the second one is formally the same. The domain of I^qI^p is defined to be $\{\sigma \in \mathfrak{D}(I^p) \colon I^p\sigma \in \mathfrak{D}(I^q)\}$. It follows from Propositions 2.1 and 2.3 (i) that

$$\begin{aligned}
\sigma \in \mathfrak{D}(I^qI^p) \quad &\Leftrightarrow \quad \sigma \in \mathfrak{M}_\infty^{p-1}(\mathbb{R}), \quad I^p\sigma \in \mathfrak{M}_\infty^{q-1}(\mathbb{R})\\
&\Leftrightarrow \quad \sigma \in \mathfrak{M}_\infty^{p+q-1}(\mathbb{R})
\end{aligned}$$

$$\Leftrightarrow \quad \sigma \in \mathfrak{D}(I^{p+q}).$$

If $\sigma \in \mathfrak{M}_\infty^{p+q-1}(\mathbb{R})$, then Lemma 2.5 shows that $(I^q(I^p\sigma))(du) = (I^{p+q}\sigma)(du)$. $\square$

Corollary 2.6. *Let $0 < q < p$. If a function f is monotone of order p on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$], then f is monotone of order q on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$].*

2.2 One-to-one property

We will prove an important result that I^p and I_+^p are one-to-one. We prepare auxiliary mappings D^q and D_+^q and two lemmas, suggested by Kamimura [15].

Definition 2.7. *Let $0 < q < 1$. Let $\mathfrak{D}(D^q)$ [resp. $\mathfrak{D}(D_+^q)$] be the class of locally finite measures ρ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] absolutely continuous with density $g(s)$ such that*

$$\int_r^\infty (s-r)^{-q-1}|g(s)-g(r)|ds < \infty \quad \text{for a. e. } r \in \mathbb{R} \text{ [resp. } \mathbb{R}_+^\circ] \tag{2.5}$$

and that the signed measure $\widetilde{\rho}$ defined by

$$\widetilde{\rho}(dr) = \left(qc_{1-q}\int_r^\infty (s-r)^{-q-1}(g(s)-g(r))ds\right)dr \tag{2.6}$$

has locally finite variation on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Define

$$D^q\rho = \widetilde{\rho} \quad [\text{resp. } D_+^q\rho = \widetilde{\rho}] \tag{2.7}$$

for $\rho \in \mathfrak{D}(D^q)$ [resp. $\mathfrak{D}(D_+^q)$].

The reason for introducing the mappings D^q and D_+^q is seen from the following lemma.

Lemma 2.8. *Let $0 < q < p < 1$ and let $\sigma \in \mathfrak{D}(I^p)$ [resp. $\mathfrak{D}(I_+^p)$]. Then $I^p\sigma \in \mathfrak{D}(D^q)$ [resp. $I_+^p\sigma \in \mathfrak{D}(D_+^p)$] and*

$$(D^qI^p\sigma)(dr) = \frac{\Gamma_{p-q}}{\Gamma_p\Gamma_{1-q}}(qC_{p,q}-1)\,(I^{p-q}\sigma)(dr) \tag{2.8}$$

[resp. the same equality with D_+^q, I_+^p, and I_+^{p-q} in place of D^q, I^p, and I^{p-q}], where

$$C_{p,q} = \int_0^1 (1-u)^{-q-1}(u^{p-1}-1)du. \tag{2.9}$$

Proof. Let $\rho = I^p\sigma$ [resp. $I_+^p\sigma$]. Then $\rho(ds) = g(s)ds$ with $g(s) = c_p\int_{(s,\infty)}(u-s)^{p-1}\sigma(du)$. For $s > r$ we have

$$g(s) - g(r)$$

$$= -c_p \int_{(r,s]} (u-r)^{p-1}\sigma(du) + c_p \int_{(s,\infty)} ((u-s)^{p-1} - (u-r)^{p-1})\sigma(du)$$
$$= -c_p \int_{(r,s]} (u-r)^{p-1}\sigma(du) + (1-p)c_p \int_{(s,\infty)} \sigma(du) \int_r^s (u-v)^{p-2} dv.$$

Let

$$J_1 = \int_r^\infty (s-r)^{-q-1} ds \int_{(r,s]} (u-r)^{p-1}\sigma(du),$$
$$J_2 = (1-p) \int_r^\infty (s-r)^{-q-1} ds \int_{(s,\infty)} \sigma(du) \int_r^s (u-v)^{p-2} dv.$$

Then

$$\int_r^\infty (s-r)^{-q-1} |g(s) - g(r)| ds \le c_p (J_1 + J_2).$$

Since $\sigma \in \mathfrak{D}(I^{p-q})$ [resp. $\mathfrak{D}(I_+^{p-q})$], we have

$$\begin{aligned}
J_1 &= \int_{(r,\infty)} (u-r)^{p-1}\sigma(du) \int_u^\infty (s-r)^{-q-1} ds \\
&= q^{-1} \int_{(r,\infty)} (u-r)^{p-q-1}\sigma(du) < \infty \quad \text{for a. e. } r \in \mathbb{R} \text{ [resp. } \mathbb{R}_+^\circ\text{]}, \\
J_2 &= (1-p) \int_{(r,\infty)} \sigma(du) \int_r^u (u-v)^{p-2} dv \int_v^u (s-r)^{-q-1} ds \\
&= (1-p) \int_{(r,\infty)} \sigma(du) \int_r^u (u-v)^{p-1} dv \int_0^1 (u-r-t(u-v))^{-q-1} dt \\
&= (1-p) \int_{(r,\infty)} \sigma(du) \int_0^1 dt \int_r^u (u-v)^{p-1} (u-r-t(u-v))^{-q-1} dv \\
&= (1-p) \int_{(r,\infty)} \sigma(du) \int_0^1 t^{-p} dt \int_0^{t(u-r)} w^{p-1} (u-r-w)^{-q-1} dw \\
&= (1-p) \int_{(r,\infty)} (u-r)^{p-q-1}\sigma(du) \int_0^1 t^{-p} dt \int_0^t x^{p-1} (1-x)^{-q-1} dx \\
&= (1-p) \int_{(r,\infty)} (u-r)^{p-q-1}\sigma(du) \int_0^1 x^{p-1} (1-x)^{-q-1} dx \int_x^1 t^{-p} dt \\
&= \widetilde{C}_{p,q} \int_{(r,\infty)} (u-r)^{p-q-1}\sigma(du) < \infty \quad \text{for a. e. } r \in \mathbb{R} \text{ [resp. } \mathbb{R}_+^\circ\text{]},
\end{aligned}$$

where

$$\widetilde{C}_{p,q} = \int_0^1 x^{p-1} (1-x)^{-q-1} (1 - x^{1-p}) dx = C_{p,q}$$

and the finiteness of $C_{p,q}$ is clear since $(1-x)^{-q-1}(1-x^{1-p}) \sim (1-p)(1-x)^{-q}$ as $x \uparrow 1$. We have thus shown (2.5) and

$$\int_r^\infty (s-r)^{-q-1} (g(s) - g(r)) ds = c_p (J_2 - J_1)$$

$$= c_p(C_{p,q} - q^{-1}) \int_{(r,\infty)} (u-r)^{p-q-1} \sigma(du).$$

Hence $I^p\sigma \in \mathfrak{D}(D^q)$ [resp. $I_+^p\sigma \in \mathfrak{D}(D_+^q)$] and

$$\begin{aligned}(D^p I^p \sigma)(dr) &= \left(c_{1-q} c_p (q C_{p,q} - 1) \int_{(r,\infty)} (u-r)^{p-q-1} \sigma(du) \right) dr \\ &= \Gamma_{p-q} c_{1-q} c_p (q C_{p,q} - 1) I^{p-q} \sigma(dr)\end{aligned}$$

on $\mathbb{R}$, and similarly on $\mathbb{R}_+^\circ$ □

Lemma 2.9. *Let $p > 0$ and let $\sigma \in \mathfrak{D}(I^p)$ [resp. $\mathfrak{D}(I_+^p)$]. Then,*

$$I^q\sigma \to \sigma \quad \textit{vaguely on } \mathbb{R} \qquad [\textit{resp. } I_+^q\sigma \to \sigma \quad \textit{vaguely on } \mathbb{R}_+^\circ] \tag{2.10}$$

as $q \downarrow 0$, that is, for all continuous functions f with compact support in $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$],

$$\int f(s) I^q\sigma(ds) \to \int f(s)\sigma(ds) \qquad [\textit{resp. } \int f(s) I_+^q\sigma(ds) \to \int f(s)\sigma(ds)] \tag{2.11}$$

as $q \downarrow 0$.

Proof. We give the proof in the case $\mathbb{R}$, but the case $\mathbb{R}_+^\circ$ is similar. First recall that $\sigma \in \mathfrak{D}(I^p)$ implies $\sigma \in \mathfrak{D}(I^q)$ for $0 < q \le p$. Assume that f is nonnegative, continuous with support in $[a,b]$ for some $a < b$. It is enough to show (2.11) for such f. Notice that

$$\int_{\mathbb{R}} f(s) I^q\sigma(ds) = \int_{\mathbb{R}} f(r)dr \int_{(r,\infty)} c_q (s-r)^{q-1} \sigma(ds) = \int_{\mathbb{R}} g_q(s)\sigma(ds),$$

where

$$g_q(s) = \int_{-\infty}^{s} c_q (s-r)^{q-1} f(r)dr.$$

We claim that

$$g_q(s) \to f(s), \qquad q \downarrow 0 \tag{2.12}$$

for $s \in \mathbb{R}$. We have $g_q(s) = 0 = f(s)$ for $s \le a$. Fix $s > a$. Let q be such that $a < s - q < s$. Then, as $q \downarrow 0$,

$$\begin{aligned}|g_q(s) - f(s)| &\le c_q \int_a^{s-q} (s-r)^{q-1} f(r)dr + c_q \int_{s-q}^{s} (s-r)^{q-1} |f(r) - f(s)|dr \\ &\quad + \left| c_q \int_{s-q}^{s} (s-r)^{q-1} dr - 1 \right| f(s) \\ &= J_1 + J_2 + J_3, \\ J_1 &\le c_q ||f|| \int_a^{s-q} (s-r)^{q-1} dr = c_{q+1} ||f|| ((s-a)^q - q^q) \to 0,\end{aligned}$$

where $||f|| = \max_{s\in\mathbb{R}} f(s)$,

$$J_2 \le \max_{r\in[s-q,s]} |f(r)-f(s)|c_{q+1}q^q \to 0,$$
$$J_3 = |c_{q+1}q^q - 1|\, f(s) \to 0.$$

This proves (2.12). If $s > a$, then

$$g_q(s) \le c_q||f|| \int_a^s (s-r)^{q-1}dr = c_{q+1}||f||(s-a)^q \le \mathrm{const}\,((s-a)\vee 1)^p$$

for $0 < q \le p$. If $s > b+1$, then

$$g_q(s) \le c_q||f|| \int_a^b (s-r)^{q-1}dr \le c_q||f||(b-a)(s-b)^{q-1} \le \mathrm{const}\,(s-b)^{p-1}$$

for $0 < q \le p$. Now, since $\sigma \in \mathfrak{M}_\infty^{p-1}(\mathbb{R})$, we can use the dominated convergence theorem and obtain

$$\int_{\mathbb{R}} g_q(s)\sigma(ds) \to \int_{\mathbb{R}} f(s)\sigma(ds), \qquad q \downarrow 0,$$

completing the proof. □

Theorem 2.10. *For any $p > 0$, I^p and I_+^p are one-to-one.*

Proof. Assume that $p < 1$. Suppose that $\sigma_1, \sigma_2 \in \mathfrak{D}(I^p)$ satisfy $I^p\sigma_1 = I^p\sigma_2$. Let $0 < q < p$. By virtue of Lemma 2.8, $I^p\sigma_j \in \mathfrak{D}(D^q)$ for $j = 1,2$ and (2.8) holds for $\sigma = \sigma_1, \sigma_2$. We have $D^qI^p\sigma_1 = D^qI^p\sigma_2$. If $qC_{p,q} - 1 \ne 0$, then it follows that $I^{p-q}\sigma_1 = I^{p-q}\sigma_2$. From the definition (2.9), $C_{p,q}$ is positive and strictly increasing with respect to q. Hence, either $qC_{p,q} - 1 \ne 0$ for all $q \in (0,p)$ or there is $q_0 \in (0,p)$ such that $qC_{p,q} - 1 \ne 0$ for all $q \in (0,p)\setminus\{q_0\}$. Thus

$$qC_{p,q} - 1 \ne 0 \quad \text{for all } q \in (0,p) \text{ sufficiently close to } p. \tag{2.13}$$

Hence

$$I^{p-q}\sigma_1 = I^{p-q}\sigma_2 \quad \text{for all } q \in (0,p) \text{ sufficiently close to } p.$$

Now, letting $q \uparrow p$ and using Lemma 2.9, we obtain $\sigma_1 = \sigma_2$. It follows that I^p is one-to-one for $0 < p < 1$. Now, using Proposition 2.4, we see that I^p is one-to-one if $p = np'$ with a positive integer n and $0 < p' < 1$. Hence I^p is one-to-one for any $p > 0$. The proof for I_+^p is similar. □

2.3 More properties and examples

When p is a positive integer, we have the following characterization. This is a result of Williamson [56]. It is given also in Lemmas 3.2 and 3.4 of Sato [37] based on Widder's book [55].

Proposition 2.11. (i) *A function $f(r)$ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] is monotone of order* 1 *if and only if it is decreasing and right-continuous on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] and tends to* 0 *as $r \to \infty$.*

(ii) *Let n be an integer ≥ 2. A function f on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] is monotone of order n if and only if*

$$\begin{cases} f(r) \text{ tends to } 0 \text{ as } r \to \infty \text{ and is } n-2 \text{ times differentiable on } \mathbb{R} \\ [\text{resp. } \mathbb{R}_+^\circ] \text{ with } (-1)^j f^{(j)} \geq 0 \text{ for } j = 0, 1, \dots, n-2, \text{ and with} \\ (-1)^{n-2} f^{(n-2)} \text{ being decreasing and convex.} \end{cases} \tag{2.14}$$

Corollary 2.12. *Let n be an integer ≥ 1. Suppose that f is n times differentiable on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Then f is monotone of order n if and only if $(-1)^j f^{(j)} \geq 0$ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] for $j = 0, 1, \dots, n$, and $f(r) \to 0$ as $r \to \infty$.*

Thus the concept of complete monotonicity of f on $\mathbb{R}_+^\circ$ coincides with that in Widder [55] and Feller [8] except the condition that $\lim_{r\to\infty} f(r) = 0$. Integral representation of a completely monotone function on $\mathbb{R}_+^\circ$ (as the Laplace transform of a measure on $\mathbb{R}_+^\circ$) is obtained from Bernstein's theorem. A completely monotone function on $\mathbb{R}$ is also represented by the Laplace transform of a measure on $\mathbb{R}_+^\circ$.

Proof of Proposition 2.11. In this proof we consider the case $\mathbb{R}$. In the case $\mathbb{R}_+^\circ$, replace $\mathbb{R}$ by $\mathbb{R}_+^\circ$.

(i) Recall that f is monotone of order 1 on $\mathbb{R}$ if and only if $f(r) = \int_{(r,\infty)} \sigma(ds)$ for some $\sigma \in \mathfrak{M}_\infty^0(\mathbb{R})$, hence if and only if $f(r)$ is finite, decreasing, and right-continuous on $\mathbb{R}$ and tends to 0 as $r \to \infty$.

(ii) Let $n \geq 2$. A function f is monotone of order n on $\mathbb{R}$ if and only if, for some $\sigma \in \mathfrak{M}_\infty^{n-1}(\mathbb{R})$,

$$\begin{aligned} f(r) &= \int_{(r,\infty)} \frac{1}{(n-1)!}(s-r)^{n-1}\sigma(ds) = \int_{(r,\infty)} \sigma(ds) \int_r^s \frac{1}{(n-2)!}(s-u)^{n-2} du \\ &= \int_{(r,\infty)} du \int_{(u,\infty)} \frac{1}{(n-2)!}(s-u)^{n-2}\sigma(ds). \end{aligned}$$

If f is monotone of order n on $\mathbb{R}$, then $f(r) \to 0$ as $r \to \infty$, since f is monotone of order 1 on $\mathbb{R}$. If f is monotone of order 2 on $\mathbb{R}$, then

$$f(r) = \int_{(r,\infty)} \sigma((u,\infty))du \tag{2.15}$$

and hence f is decreasing and convex. Conversely, if $f(r)$ is decreasing, convex, and convergent to 0 as $r \to \infty$, then f is written as in (2.15) with some $\sigma \in \mathfrak{M}_\infty^1(\mathbb{R})$ and hence f is monotone of order 2 on $\mathbb{R}$.

Now let $n \geq 3$ and suppose that assertion (ii) is true with $n-1$ in place of n. If f is monotone of order n on $\mathbb{R}$, then $g(u) = \int_{(u,\infty)} \frac{1}{(n-2)!}(s-u)^{n-2}\sigma(ds)$ is monotone of order $n-1$ on $\mathbb{R}$ and, a fortiori, continuous and hence $-f'(r) = g(r)$, which shows that (2.14) is satisfied. Conversely, suppose that f satisfies (2.14). Then $-f'(r) \to 0$ as $r \to \infty$, since otherwise $f(r)$ goes to $-\infty$ as $r \to \infty$. Hence (2.14) is satisfied with

$-f'$ in place of f and with $n-1$ in place of n. Hence

$$-f'(u) = \int_{(u,\infty)} \frac{1}{(n-2)!}(s-u)^{n-2}\sigma(ds)$$

for some $\sigma \in \mathfrak{M}_\infty^{n-2}(\mathbb{R})$. Since $-f'(u)$ is continuous and since $f(r) \to 0$ as $r \to \infty$, we have $f(r) = \int_r^\infty f'(u)du$ and hence

$$f(r) = \int_{(r,\infty)} \frac{1}{(n-1)!}(s-r)^{n-1}\sigma(ds).$$

As (2.14) implies that f is locally integrable on $\mathbb{R}$, σ belongs to $\mathfrak{D}(I^n)$ and f is monotone of order n on $\mathbb{R}$. $\square$

Let us give some necessary conditions for f to be monotone of order p.

Proposition 2.13. *Suppose that f is monotone of order p on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] for some $p>0$ and that f is not identically zero. Then:*

(i) *f is lower semi-continuous on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$].*

(ii) *Either $f(r)>0$ for all $r \in \mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] or there is $a \in \mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] such that $f(r)>0$ for $r<a$ and $f(r)=0$ for $r \geq a$.*

(iii) *In the case of $\mathbb{R}$, $\liminf_{r\to-\infty}(f(r)/|r|^{p-1})>0$.*

(iv) *In the case of $\mathbb{R}_+^\circ$, $\liminf_{r\downarrow 0} f(r)>0$.*

Proof. The function f satisfies (1.10) for some $\sigma \in \mathfrak{M}_\infty^{p-1}(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$] with $\sigma \neq 0$.

(i) Using Fatou's lemma, we see

$$\begin{aligned}\liminf_{r'\to r} f(r') &\geq c_p \int \liminf_{r'\to r}(1_{(r',\infty)}(s)(s-r')^{p-1})\sigma(ds)\\ &= c_p \int_{(r,\infty)} (s-r)^{p-1}\sigma(ds) = f(r),\end{aligned}$$

that is, f is lower semi-continuous.

(ii) If $f(r_0)>0$ for some r_0, then $f(r)>0$ for all $r \leq r_0$, because $\int_{(r_0,\infty)}(s-r_0)^{p-1}\sigma(ds)>0$ shows that there is a point s_0 in the support of σ such that $s_0>r_0$.

(iii) Choose $-\infty<a<b<\infty$ such that $\sigma((a,b))>0$. Let $r<a$. Then

$$f(r) \geq c_p \int_{(a,b)} (s-r)^{p-1}\sigma(ds) \geq \begin{cases} c_p(b-r)^{p-1}\sigma((a,b)) & \text{if } p \leq 1,\\ c_p(a-r)^{p-1}\sigma((a,b)) & \text{if } p>1.\end{cases}$$

Hence the assertion follows.

(iv) Proved similarly to (iii). $\square$

Proposition 2.14. *Suppose that f is monotone of order p on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] for some $p>1$. Then f is absolutely continuous on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$].*

Proof. Consider the case of $\mathbb{R}$. We have (1.10) for f with some $\sigma \in \mathfrak{D}(I^p)$. Since $I^p = I^1 I^{p-1}$, it follows from Lemma 2.5 that

$$f(r) = \int_{(r,\infty)} (I^{p-1}\sigma)(ds) = \int_r^\infty g(s)ds$$

for some $g(s) \geq 0$. The case of $\mathbb{R}_+^\circ$ is similar. $\square$

Let $S = S^{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. This is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ if $d \geq 2$ and the two-point set $\{-1,1\}$ if $d = 1$. A family $\{\sigma_\xi : \xi \in S\}$ of locally finite measures on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] is called a *measurable family* if $\sigma_\xi(E)$ is measurable in $\xi \in S$ for every $E \in \mathscr{B}(\mathbb{R})$ [resp. $\mathscr{B}(\mathbb{R}_+^\circ)$]. If $\{\sigma_\xi : \xi \in S\}$ is a measurable family, then, (a) for any $[0,\infty]$-valued function $f(r,s)$ measurable in (r,s), $\int f(r,s)\sigma_\xi(ds)$ is measurable in (ξ,r), and (b) for any $a > 0$, $\sigma_\xi((r,r+a])$ is measurable in (ξ,r). To see (a), use the monotone class theorem. To see (b), apply (a) to $f(r,s) = 1_{(r,r+a]}(s)$.

Proposition 2.15. *Let $p > 0$. If $\{\sigma_\xi : \xi \in S\}$ is a measurable family of measures in $\mathfrak{M}_\infty^{p-1}(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$], then $\{I^p(\sigma_\xi) : \xi \in S\}$ [resp. $\{I_+^p(\sigma_\xi) : \xi \in S\}$] is a measurable family.*

Proof. Notice that, for any $E \in \mathscr{B}(\mathbb{R})$

$$\begin{aligned} I^p(\sigma_\xi)(E) &= \int_E dr \int_{(r,\infty)} c_p(s-r)^{p-1}\sigma_\xi(ds) \\ &= \int_{\mathbb{R}} \sigma_\xi(ds) \int_{E\cap(-\infty,s)} c_p(s-r)^{p-1}dr, \end{aligned}$$

which is measurable in ξ. The case of $\mathbb{R}_+^\circ$ is similar. $\square$

Proposition 2.16. *Let $p > 0$ and let $\{\sigma_\xi : \xi \in S\} \subset \mathfrak{M}_\infty^{p-1}(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$]. If $\{I^p(\sigma_\xi) : \xi \in S\}$ [resp. $\{I_+^p(\sigma_\xi) : \xi \in S\}$] is a measurable family, then $\{\sigma_\xi : \xi \in S\}$ is a measurable family.*

Proof. Consider the case of $\mathbb{R}$. The case of $\mathbb{R}_+^\circ$ is similar. Let $\{I^p(\sigma_\xi)\}$ be a measurable family. For each ξ

$$I^p(\sigma_\xi)(E) = \int_E g_\xi(r)dr, \quad g_\xi(r) = \int_{(r,\infty)} c_p(s-r)^{p-1}\sigma_\xi(ds).$$

Let

$$\widetilde{g}_\xi(r) = \liminf_{n\to\infty} n\int_r^{r+1/n} g_\xi(r')dr' = \liminf_{n\to\infty} nI^p(\sigma_\xi)((r,r+1/n]).$$

Then $\widetilde{g}_\xi(r)$ is measurable in (ξ,r) and, by Lebesgue's differentiation theorem, $g_\xi(r) = \widetilde{g}_\xi(r)$ for a. e. s for every fixed ξ. Thus $I^p(\sigma_\xi)(dr) = \widetilde{g}_\xi(r)dr$.

Suppose $0 < p < 1$. Let $0 < q < p$. Then $\{D^q I^p(\sigma_\xi) : \xi \in S\}$ is a measurable family. It follows from Lemma 2.8 and (2.13) that $\{I^{p-q}(\sigma_\xi) : \xi \in S\}$ is a measurable family for q sufficiently close to p. Hence, by Lemma 2.9, $\{\sigma_\xi : \xi \in S\}$ is a measurable family. Now, for any $p > 0$, write $p = np'$ with positive integer n and $0 < p' < 1$ and use Proposition 2.4 to see $\{\sigma_\xi : \xi \in S\}$ is a measurable family. $\square$

Example 2.17. Let $p>0$. In the following, σ is in $\mathfrak{M}_\infty^{p-1}(\mathbb{R})$ or in $\mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$ and we write

$$f_p(r)=c_p\int_{(r,\infty)}(s-r)^{p-1}\sigma(ds) \tag{2.16}$$

for $r\in\mathbb{R}$ or for $r\in\mathbb{R}_+^\circ$. Thus f_p is monotone of order p on $\mathbb{R}$ or on $\mathbb{R}_+^\circ$.

(a) A δ-distribution located at x is denoted by δ_x. Let $\sigma=\delta_a$ with $a\in\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Then

$$f_p(r)=\begin{cases} c_p(a-r)^{p-1}, & r<a,\\ 0, & r\geq a.\end{cases}$$

Hence $f_p(r)$ is strictly increasing for $r<a$ if $p<1$; f_p equals 1 for $r<a$ if $p=1$; f_p is not continuous if $p\leq 1$. If $p>1$, then f_p is strictly decreasing for $r<a$ and continuous on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. For any $p'>p$, f_p is not monotone of order p'. Indeed, otherwise Proposition 2.4 and Theorem 2.10 show that $\delta_a=I^{p'-p}\tau$ [resp. $I_+^{p'-p}\tau$] for some $\tau\in\mathfrak{M}_\infty^{p'-1}(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^{p'-1}(\mathbb{R}_+^\circ)$], which is absurd since $I^{p'-p}\tau$ [resp. $I_+^{p'-p}\tau$] is absolutely continuous.

Notice that this function $f_p(r)$ has the following property: if $\alpha\in\mathbb{R}$ satisfies $\alpha(p-1)>-1$, then $f_p(r)^\alpha$ is monotone of order $\alpha(p-1)+1$ and not monotone of order p' for any $p'>\alpha(p-1)+1$.

(b) Let $-\infty<a<b<\infty$ [resp. $0<a<b<\infty$] and let $\sigma(ds)=1_{(a,b]}(s)ds$. Then

$$f_p(r)=\begin{cases} c_{p+1}((b-r)^p-(a-r)^p), & r<a,\\ c_{p+1}(b-r)^p, & a\leq r<b,\\ 0, & r\geq b.\end{cases}$$

Thus

$$f_p'(r)=c_p((a-r)^{p-1}-(b-r)^{p-1})\quad \text{for } r<a.$$

Hence, if $p<1$, then f_p is strictly increasing for $r\leq a$ and strictly decreasing for $a\leq r\leq b$. For all $p>0$, f_p is continuous on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. For any $p'>p$, f_p is not monotone of order p' on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Indeed, otherwise $\sigma=I^{p'-p}\tau$ [resp. $I_+^{p'-p}\tau$] for some $\tau\in\mathfrak{M}_\infty^{p'-1}(\mathbb{R})$ [resp. $\mathfrak{M}_\infty^{p'-1}(\mathbb{R}_+^\circ)$], which contradicts Proposition 2.13.

(c) Let $\sigma(ds)=s^{-\alpha}ds$ on $\mathbb{R}_+^\circ$ with $\alpha>p$. Then $\sigma\in\mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$ and the function f_p is monotone of order p on $\mathbb{R}_+^\circ$ and

$$f_p(r)=c_p\int_r^\infty(s-r)^{p-1}s^{-\alpha}ds=c_pr^{p-\alpha}\int_1^\infty(u-1)^{p-1}u^{-\alpha}du=c_+r^{p-\alpha}$$

for $r>0$, where

$$c_+=c_p\int_0^\infty u^{p-1}(u+1)^{-\alpha}du=c_pB(p,\alpha-p)=\Gamma_{\alpha-p}/\Gamma_\alpha.$$

(d) Suppose $0<p<1$. Let $\sigma(ds)=(s-b)^{-\alpha}1_{(b,\infty)}(s)ds$ on $\mathbb{R}$ with $1>\alpha>p$ and $b\in\mathbb{R}$. Then $\sigma\in\mathfrak{M}_\infty^{p-1}(\mathbb{R})$ and f_p is monotone of order p on $\mathbb{R}$ and

$$f_p(r) = \begin{cases} c_-(b-r)^{p-\alpha}, & r < b \\ \infty, & r = b \\ c_+(r-b)^{p-\alpha}, & r > b, \end{cases}$$

where c_+ is the same as in (c) and

$$c_- = c_p \int_0^\infty (u+1)^{p-1} u^{-\alpha} du = c_p B(1-\alpha, \alpha-p) = \Gamma_{1-\alpha}\Gamma_{\alpha-p}/(\Gamma_p\Gamma_{1-p}).$$

Note that $f_p(b) = c_p \int_b^\infty (s-b)^{p-\alpha-1} ds = \infty$. This f_p is a $(0,\infty]$-valued continuous function on $\mathbb{R}$, strictly increasing on $(-\infty, b)$, equal to ∞ at b, and strictly decreasing on (b,∞). For any $p' > p$, this f_p is not monotone of order p' by the same reason as in (b).

(e) Let $0 < p < \alpha < 1$. Let $B = \{b_1, b_2, \ldots\}$ be a countable set in $\mathbb{R}$. Choose $C_n > 0$, $n = 1, 2, \ldots$, satisfying

$$\sum_{b_n \in B \cap (-\infty,1]} C_n + \sum_{b_n \in B \cap (1,\infty)} C_n b_n^{p-\alpha} < \infty.$$

Let

$$\sigma(ds) = \sum_{n=1}^\infty C_n (s-b_n)^{-\alpha} 1_{(b_n,\infty)}(s) ds.$$

Then $\sigma \in \mathfrak{M}_\infty^{p-1}(\mathbb{R})$, since we have

$$\int_{(1,\infty)} s^{p-1} \sigma(ds) = \sum_{n=1}^\infty C_n \int_{b_n \vee 1}^\infty s^{p-1} (s-b_n)^{-\alpha} ds < \infty,$$

noting that, for $b_n \le 1$,

$$\begin{aligned} \int_1^\infty s^{p-1}(s-b_n)^{-\alpha} ds &\le \int_1^\infty s^{p-1}(s-1)^{-\alpha} ds = \int_0^\infty (u+1)^{p-1} u^{-\alpha} du \\ &= B(1-\alpha, \alpha-p) \end{aligned}$$

and, for $b_n > 1$,

$$\int_{b_n}^\infty s^{p-1}(s-b_n)^{-\alpha} ds = b_n^{p-\alpha} \int_1^\infty u^{p-1}(u-1)^{-\alpha} du = b_n^{p-\alpha} B(1-\alpha, \alpha-p).$$

Let $f_{p,\alpha,b}(s)$ denote the function in (d). Then

$$f_p(r) = \sum_{n=1}^\infty C_n f_{p,\alpha,b_n}(r)$$

and $f_p(b_n) = \infty$ for $n = 1, 2, \ldots$. If the set B has supremum ∞, then $\limsup_{s\to\infty} f_p(s) = \infty$. If B is a dense set in $\mathbb{R}$, then $f_p(s)$ is finite almost everywhere but infinite on the dense set. □

Example 2.18. (a) Let $f(r) = r^{-\beta}$ for $r > 0$ with $\beta > 0$. Then f is completely monotone on $\mathbb{R}_+^\circ$, because, for any $p > 0$, we can choose $\alpha = p + \beta$ and apply Example 2.17 (c). Alternatively, use Proposition 2.11.

(b) Let $f(r) = e^{-r}$ for $r \in \mathbb{R}$. Then f is completely monotone on $\mathbb{R}$. Use Proposition 2.11 or $c_p \int_r^\infty (s-r)^{p-1} e^{-s} ds = c_p \int_0^\infty u^{p-1} e^{-u-r} du = e^{-r}$.

(c) Let

$$f(r) = \begin{cases} \arcsin(1-r), & 0 < r < 1, \\ 0, & r \geq 1. \end{cases}$$

Then f is monotone of order 2 on $\mathbb{R}_+^\circ$, since it is decreasing and convex. For any $p > 2$, f is not monotone of order p on $\mathbb{R}_+^\circ$. To prove this, suppose f is monotone of order $p > 2$ on $\mathbb{R}_+^\circ$. Then $f(r)dr = I_+^p \sigma$ for some $\sigma \in \mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$. Hence $f(r)dr = I_+^1 \tau$ with $\tau = I_+^{p-1}\sigma$. On the other hand

$$f(r) = \int_r^\infty g(s) ds \quad \text{with} \quad g(s) = (1-(1-s)^2)^{-1/2} 1_{(0,1)}(s).$$

Hence $\tau(ds) = g(s)ds$ by Theorem 2.10. Hence $g(s)$ is equal almost everywhere on $\mathbb{R}_+^\circ$ to a function monotone of order $p-1$. Since $p - 1 > 1$, it follows that $g(s)$ is equal almost everywhere on $\mathbb{R}_+^\circ$ to an absolutely continuous function (Proposition 2.14). This is absurd.

(d) Let

$$f(r) = \begin{cases} -\log r, & 0 < r < 1, \\ 0, & r \geq 1. \end{cases}$$

Then, similarly to the previous example, f is monotone of order 2 on $\mathbb{R}_+^\circ$ but is not monotone of order p on $\mathbb{R}_+^\circ$ for any $p > 2$. □

Example 2.19. Let $g(r) = \sqrt{r^2+1} - r$, $r \in \mathbb{R}$, and $h_\alpha(r) = g(r)^\alpha$, $r \in \mathbb{R}$, with $\alpha \in (0,\infty)$. The function g is monotone of order 2 on $\mathbb{R}$, since $g(r) > 0$, $-g'(r) = 1 - r(r^2+1)^{-1/2} > 0$, and

$$g''(r) = (r^2+1)^{-1/2} - r^2(r^2+1)^{-3/2} = (r^2+1)^{-3/2} > 0,$$

$$g(r) = |r|\sqrt{1+|r|^{-2}} - r = |r|(1+O(|r|^{-2})) - r = O(r^{-1}), \quad r \to \infty.$$

Let us show the following.

(a) For every $\alpha > 0$, h_α is not monotone of order p on $\mathbb{R}$ for any $p > \alpha + 1$.

(b) For every $\alpha > 0$, h_α is monotone of order 1 on $\mathbb{R}$.

(c) The following statement is true for $n = 1,2,3$. For any $\alpha \geq n$, h_α is monotone of order $n+1$ on $\mathbb{R}$.

We have $g(r) = 2|r| + O(|r|^{-1})$, $r \to -\infty$. Hence we see (a) by virtue of Proposition 2.13 (iii), because $h_\alpha(r)/|r|^{p-1} \sim 2^\alpha/|r|^{p-\alpha-1}$ as $r \to -\infty$. We have (b), since

$$h'_\alpha = \frac{\alpha(\sqrt{r^2+1} - r)^{\alpha-1}}{\sqrt{r^2+1}}(r - \sqrt{r^2+1}) = \frac{-\alpha h_\alpha}{\sqrt{r^2+1}}, \tag{2.17}$$

which is negative on $\mathbb{R}$. We have (c) for $n=1$, since

$$h''_\alpha = \alpha\left(\frac{rh_\alpha}{(r^2+1)^{3/2}} - \frac{h'_\alpha}{\sqrt{r^2+1}}\right) = \frac{\alpha h_\alpha}{(r^2+1)^{3/2}}(r+\alpha\sqrt{r^2+1}), \tag{2.18}$$

which is positive on $\mathbb{R}$ for $\alpha \ge 1$.

The following recursion formula is known for the derivatives of h_α ([30] p. 41):

$$(r^2+1)h_\alpha^{(j+2)} + (2j+1)rh_\alpha^{(j+1)} + (j^2-\alpha^2)h_\alpha^{(j)} = 0. \tag{2.19}$$

Indeed, this is true for $j=0$ from (2.17) and (2.18); if (2.19) is true for a given $j\ge 0$, then its differentiation shows that it is true with $j+1$ in place of j.

Now let us prove (c) for $n=2$. It follows from (2.17), (2.18), and (2.19) that

$$\begin{aligned}
(r^2+1)h'''_\alpha &= -3rh''_\alpha - (1-\alpha^2)h'_\alpha = \frac{-3\alpha r h_\alpha}{(r^2+1)^{3/2}}(r+\alpha\sqrt{r^2+1}) + \frac{(1-\alpha^2)\alpha h_\alpha}{\sqrt{r^2+1}} \\
&= \frac{-\alpha h_\alpha}{(r^2+1)^{3/2}}(3\alpha r\sqrt{r^2+1} + (\alpha^2+2)r^2 + (\alpha^2-1)) \\
&= \frac{-\alpha h_\alpha}{(r^2+1)^{3/2}}[\tfrac{3}{2}\alpha(\sqrt{r^2+1}+r)^2 + (\alpha-2)(\alpha-1)r^2 + (\alpha-2)(\alpha+\tfrac{1}{2})],
\end{aligned}$$

which is negative on $\mathbb{R}$ for $\alpha \ge 2$.

Let us prove (c) for $n=3$. We have

$$\begin{aligned}
(r^2+1)h_\alpha^{(4)} &= -5rh'''_\alpha - (4-\alpha^2)h''_\alpha \\
&= \frac{5\alpha r h_\alpha}{(r^2+1)^{5/2}}(3\alpha r\sqrt{r^2+1} + (\alpha^2+2)r^2 + (\alpha^2-1)) \\
&\quad - (4-\alpha^2)\frac{\alpha h_\alpha}{(r^2+1)^{3/2}}(r+\alpha\sqrt{r^2+1}) \\
&= \frac{\alpha h_\alpha}{(r^2+1)^{5/2}}[(\alpha^2+11)\alpha r^2\sqrt{r^2+1} + (\alpha^2-4)\alpha\sqrt{r^2+1} \\
&\quad + 6(\alpha^2+1)r^3 + 3(2\alpha^2-3)r] \\
&= \frac{\alpha h_\alpha}{(r^2+1)^{5/2}}[\tfrac{3}{2}(\alpha^2+1)(\sqrt{r^2+1}+r)^3 + \tfrac{3}{2}(\alpha^2-9)r \\
&\quad + (\alpha^3-6\alpha^2+11\alpha-6)r^2\sqrt{r^2+1} + (\alpha^3-\tfrac{3}{2}\alpha^2-4\alpha-\tfrac{3}{2})\sqrt{r^2+1}] \\
&= \frac{\alpha h_\alpha}{(r^2+1)^{5/2}}[\tfrac{3}{2}(\alpha^2+1)(\sqrt{r^2+1}+r)^3 + \tfrac{3}{2}(\alpha^2-9)(\sqrt{r^2+1}+r) \\
&\quad + (\alpha-3)(\alpha-2)(\alpha-1)r^2\sqrt{r^2+1} + (\alpha-3)(\alpha^2-4)\sqrt{r^2+1}],
\end{aligned}$$

which is positive on $\mathbb{R}$ for $\alpha \ge 3$. This shows (c) for $n=3$. □

Remark 2.20. Open question: In the notation of Example 2.19, is h_α monotone of order $\alpha+1$ for every $\alpha > 0$? □

Some transformations more general than the fractional integral $I_+^p\sigma$ are studied by Maejima, Pérez-Abreu, and Sato [24], which is related to [23].

3 Preliminaries in probability theory

3.1 Lévy–Khintchine representation of infinitely divisible distributions

We also use a representation of the cumulant function $C_\mu(z)$ of $\mu \in ID$ other than (1.1) in the form

$$C_\mu(z) = -\frac{1}{2}\langle z, A_\mu z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2}\right)\nu_\mu(dx) + i\langle \gamma_\mu^\sharp, z\rangle. \tag{3.1}$$

Here $\gamma_\mu^\sharp$ is an element of $\mathbb{R}^d$; A_μ and ν_μ are common to (1.1) and (3.1). Throughout this article $\gamma_\mu^\sharp$ is used in this sense. It follows from (1.1) and (3.1) that

$$\gamma_\mu^\sharp = \gamma_\mu - \int_{|x|\le 1}\frac{x|x|^2}{1+|x|^2}\nu_\mu(dx) + \int_{|x|>1}\frac{x}{1+|x|^2}\nu_\mu(dx). \tag{3.2}$$

The triplets $(A_\mu, \nu_\mu, \gamma_\mu)$ and $(A_\mu, \nu_\mu, \gamma_\mu^\sharp)$ are both called the Lévy–Khintchine triplet of μ. Each has its own advantage and disadvantage. Weak convergence of a sequence of infinitely divisible distributions can be expressed by the corresponding triplets of the type $(A_\mu, \nu_\mu, \gamma_\mu^\sharp)$, but cannot by the triplets of the type $(A_\mu, \nu_\mu, \gamma_\mu)$. This is because the integrand in the integral term is continuous with respect to x in (3.1), but not continuous in (1.1). On the other hand the formulas derived from $(A_\mu, \nu_\mu, \gamma_\mu)$ are often simpler than those derived from $(A_\mu, \nu_\mu, \gamma_\mu^\sharp)$. See the book [39] for details. In [39] the author uses the symbol γ in the sense of γ_μ, but in the papers [40]–[44] in the sense of $\gamma_\mu^\sharp$.

The γ_μ and $\gamma_\mu^\sharp$ are both called the location parameter of μ. They depend on the choice of the integrand in the Lévy–Khintchine representation. Many other choices of the integrand are found in the literature. Kwapień and Woyczyński [17] and Rajput and Rosinski [31] use some form other than in (1.1) and (3.1). Maruyama [29] uses still another form.

3.2 Radial and spherical decompositions of σ-finite measures on $\mathbb{R}^d$

A measure $\nu(B)$, $B \in \mathscr{B}(\mathbb{R}^d)$, is called σ-finite if there is a Borel partition B_n, $n = 1, 2, \ldots$, of $\mathbb{R}^d$ such that $\nu(B_n) < \infty$. The following propositions give two decompositions of σ-finite measures on $\mathbb{R}^d$.

Proposition 3.1. *Let ν be a σ-finite measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$. Then there are a σ-finite measure λ on $S = \{\xi : |\xi| = 1\}$ with $\lambda(S) \geq 0$ and a measurable family $\{\nu_\xi : \xi \in S\}$ of σ-finite measures on $\mathbb{R}_+^\circ$ with $\nu_\xi(\mathbb{R}_+^\circ) > 0$ such that*

$$\nu(B) = \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi)\nu_\xi(dr), \qquad B \in \mathscr{B}(\mathbb{R}^d). \tag{3.3}$$

Here λ and ν_ξ are uniquely determined in the following sense: if $(\lambda(d\xi), \nu_\xi)$ and $(\lambda'(d\xi), \nu'_\xi)$ both satisfy these conditions, then there is a measurable function $c(\xi)$ on S such that

$$0 < c(\xi) < \infty, \tag{3.4}$$
$$c(\xi)\lambda'(d\xi) = \lambda(d\xi), \tag{3.5}$$
$$\nu'_\xi(dr) = c(\xi)\nu_\xi(dr) \quad \text{for } \lambda\text{-a. e. } \xi \in S. \tag{3.6}$$

We call the pair $(\lambda(d\xi), \nu_\xi)$ in this proposition a *radial decomposition* or *polar decomposition* of ν.

Proof of Proposition 3.1. If $\nu = 0$, then $\lambda = 0$ and arbitrary ν_ξ satisfy the assertion.

Assume that $\nu \neq 0$. Let B_n, $n = 1, 2, \ldots$, be a Borel partition of $\mathbb{R}^d \setminus \{0\}$ such that $a_n = \nu(B_n) < \infty$. If $a_n > 0$, then let $f(x) = 2^{-n}/a_n$ for $x \in B_n$. If $a_n = 0$, then let $f(x) = 1$ for $x \in B_n$. Let $b = \int_{\mathbb{R}^d \setminus \{0\}} f(x)\nu(dx)$. We have $0 < b \leq \sum_{n=1}^\infty 2^{-n}$. Let $\widetilde{\nu}(dx) = b^{-1} f(x)\nu(dx)$, which is a probability measure. Using the conditional distribution theorem, we find a probability measure $\widetilde{\lambda}$ on S and a measurable family $\{\widetilde{\nu}_\xi : \xi \in S\}$ of probability measures on $\mathbb{R}_+^\circ$ such that

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi)\widetilde{\nu}_\xi(dr), \qquad B \in \mathscr{B}(\mathbb{R}^d).$$

Thus

$$\nu(B) = \int_B b f(x)^{-1} \widetilde{\nu}(dx) = \int_S \widetilde{\lambda}(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi) b f(r\xi)^{-1} \widetilde{\nu}_\xi(dr).$$

Let $\lambda = \widetilde{\lambda}$ and $\nu_\xi(dr) = b f(r\xi)^{-1}\widetilde{\nu}_\xi(dr)$. Then ν_ξ is a σ-finite measure on $\mathbb{R}_+^\circ$ for each ξ and (3.3) holds. To see the uniqueness, let $(\lambda(d\xi), \nu_\xi)$ be the pair just constructed, and let $(\lambda'(d\xi), \nu'_\xi)$ be another decomposition of ν. Then, for every $E \in \mathscr{B}(S)$,

$$\begin{aligned}\lambda(E)=\widetilde{\lambda}(E)=\widetilde{\nu}((0,\infty)E)&=\int_{(0,\infty)E} b^{-1}f(x)\nu(dx)\\&=\int_E \lambda'(d\xi)\int_{\mathbb{R}_+^\circ} b^{-1}f(r\xi)\nu'_\xi(dr).\end{aligned}$$

Let $c(\xi)=\int_{\mathbb{R}_+^\circ} b^{-1}f(r\xi)\nu'_\xi(dr)$. Then $c(\xi)$ is positive for all ξ and finite for λ'-a. e. ξ. Modify $c(\xi)$ on a λ'-null set so that (3.4) holds. Now we have (3.5). Then (3.6) also follows. It follows that (3.4)–(3.6) hold for arbitrary two decompositions with an appropriate $c(\xi)$. □

Remark 3.2. If $\nu\neq 0$, then we can choose the measure λ in Proposition 3.1 to be a probability measure. Indeed, $\widetilde{\lambda}$ in the proof is a probability measure. □

Proposition 3.3. *Let ν be a σ-finite measure on $\mathbb{R}^d$ satisfying $\nu(\{0\})=0$. Then there are a σ-finite measure $\overline{\nu}$ on $\mathbb{R}_+^\circ$ with $\overline{\nu}(\mathbb{R}_+^\circ)\geq 0$ and a measurable family $\{\lambda_r\colon r\in\mathbb{R}_+^\circ\}$ of σ-finite measures on $S=\{\xi:|\xi|=1\}$ with $\lambda_r(S)>0$ such that*

$$\nu(B)=\int_{\mathbb{R}_+^\circ}\overline{\nu}(dr)\int_S 1_B(r\xi)\lambda_r(d\xi),\qquad B\in\mathscr{B}(\mathbb{R}^d). \tag{3.7}$$

Here $\overline{\nu}$ and λ_r are uniquely determined in the following sense: if $(\overline{\nu}(dr),\lambda_r)$ and $(\overline{\nu}'(dr),\lambda'_r)$ both satisfy these conditions, then there is a measurable function $c(r)$ on $\mathbb{R}_+^\circ$ such that

$$0<c(r)<\infty, \tag{3.8}$$

$$c(r)\overline{\nu}'(dr)=\overline{\nu}(dr), \tag{3.9}$$

$$\lambda'_r(d\xi)=c(r)\lambda_r(d\xi)\quad \textit{for } \overline{\nu}\textit{-a. e. } r\in\mathbb{R}_+^\circ. \tag{3.10}$$

Proof is similar to that of Proposition 3.1, interchanging the roles of S and $\mathbb{R}_+^\circ$. We call the pair $(\overline{\nu}(dr),\lambda_r)$ in this proposition a *spherical decomposition* of ν.

Remark 3.4. If there is a positive measurable function $f(r)$ on $\mathbb{R}_+^\circ$ such that $\int_{\mathbb{R}^d\setminus\{0\}} f(|x|)\nu(dx)<\infty$, then λ_r, $r\in S$, in Proposition 3.3 can be chosen to be probability measures. Indeed, since $\int_{\mathbb{R}^d} f(|x|)\nu(dx)=\int_{\mathbb{R}_+^\circ} f(r)\lambda_r(S)\overline{\nu}(dr)$, $\lambda_r(S)$ is finite for $\overline{\nu}$-a. e. r. Noting that

$$\nu(B)=\int_{\mathbb{R}_+^\circ}\lambda_r(S)\overline{\nu}(dr)\int_S 1_B(r\xi)(\lambda_r(S))^{-1}\lambda_r(d\xi),$$

choose $\widetilde{\nu}(dr)=\lambda_r(S)\overline{\nu}(dr)$, $\widetilde{\lambda}_r(d\xi)=(\lambda_r(S))^{-1}\lambda_r(d\xi)$ for $\overline{\nu}$-a. e. r, and $\widetilde{\lambda}_r$ appropriately for r in a $\overline{\nu}$-null set and consider $(\widetilde{\nu}(dr),\widetilde{\lambda}_r)$ as a new spherical decomposition. □

We say that the Lévy measure ν_μ of $\mu\in ID$ is *of polar product type* if there are a finite measure λ_μ on S and a σ-finite measure $\overline{\nu}_\mu$ on $\mathbb{R}_+^\circ$ such that

$$\nu(B) = \int_S \lambda_\mu(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi)\bar{\nu}_\mu(dr), \quad B \in \mathscr{B}(\mathbb{R}^d). \tag{3.11}$$

Example 3.5. Any stable distribution μ on $\mathbb{R}^d$ has Lévy measure of polar product type. Indeed, if μ is α-stable, then $\nu_\mu(B) = \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi) r^{-\alpha-1} dr$ with a finite measure λ on S. □

3.3 Weak mean of infinitely divisible distributions

As usual, a distribution μ on $\mathbb{R}^d$ is said to have mean m_μ if $\int_{\mathbb{R}^d} |x|\mu(dx) < \infty$ and $\int_{\mathbb{R}^d} x\mu(dx) = m_\mu$. We say that μ has mean in $\mathbb{R}^d$ if $\int_{\mathbb{R}^d} |x|\mu(dx) < \infty$.

Definition 3.6. *Let $\mu \in ID$. We say that μ* has weak mean in $\mathbb{R}^d$ *if*

$$\int_{1<|x|\le a} x\nu_\mu(dx) \text{ is convergent in } \mathbb{R}^d \text{ as } a \to \infty \tag{3.12}$$

We say that μ has weak mean m_μ *if* (3.12) *holds and*

$$C_\mu(z) = \frac{-1}{2}\langle z, A_\mu z\rangle + \lim_{a\to\infty} \int_{|x|\le a} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_\mu(dx) + i\langle m_\mu, z\rangle. \tag{3.13}$$

If μ satisfies (3.12)*, write*

$$m_\mu^L = \lim_{a\to\infty} \int_{1<|x|\le a} x\nu_\mu(dx). \tag{3.14}$$

Remark 3.7. Condition (3.12) is equivalent to the property that

$$\int_{1<|x|\le a} \langle z,x\rangle \nu_\mu(dx) \text{ is convergent in } \mathbb{R} \text{ as } a \to \infty, \text{ for } z \in \mathbb{R}^d,$$

because $\int_{1<|x|\le a} \langle z,x\rangle \nu_\mu(dx) = \langle z, \int_{1<|x|\le a} x\nu_\mu(dx)\rangle$. Condition (3.12) is also equivalent to the property that

$$\int_{|x|\le a} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_\mu(dx) \text{ is convergent in } \mathbb{C} \text{ as } a \to \infty, \text{ for } z \in \mathbb{R}^d.$$

Indeed,

$$\begin{aligned} &\int_{|x|\le a} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_\mu(dx) \\ &\quad = \int_{|x|\le a} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\le 1\}}(x))\nu_\mu(dx) - i\int_{1<|x|\le a} \langle z,x\rangle \nu_\mu(dx) \end{aligned} \tag{3.15}$$

for $a > 1$ and the first term in the right-hand side is always convergent as $a \to \infty$. □

Remark 3.8. Let $\mu \in ID$. Then μ has mean m_μ if and only if $\int_{|x|>1} |x| \nu_\mu(dx) < \infty$ and

$$C_\mu(z) = \frac{-1}{2}\langle z, A_\mu z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_\mu(dx) + i\langle m_\mu, z\rangle \tag{3.16}$$

(see Example 25.12 of [39]). Therefore, if μ has mean m_μ, then μ has weak mean m_μ. □

Proposition 3.9. *Let $\mu \in ID$. If μ has weak mean m_μ in $\mathbb{R}^d$, then*

$$m_\mu = m_\mu^L + \gamma_\mu. \tag{3.17}$$

Proof. If μ has weak mean in $\mathbb{R}^d$, then it follows from Remark 3.7 and (3.15) that

$$\begin{aligned}\lim_{a\to\infty} \int_{|x|\le a} &(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_\mu(dx) \\ &= \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\le 1\}}(x))\nu_\mu(dx) - i\langle m_\mu^L, z\rangle.\end{aligned}$$

Combined with (1.1), this shows that μ satisfies (3.13) with $m_\mu = m_\mu^L + \gamma_\mu$. □

We strengthen the notion of having weak mean.

Definition 3.10. *Let $\mu \in ID$. We say that μ* has weak mean in $\mathbb{R}^d$ absolutely *if*

$$\int_{(1,\infty)} r\bar{\nu}_\mu(dr) \left| \int_S \xi \lambda_r^\mu(d\xi) \right| < \infty, \tag{3.18}$$

where $(\bar{\nu}_\mu(dr), \lambda_r^\mu)$ is a spherical decomposition of ν_μ. We say that μ has weak mean m_μ absolutely *if μ has weak mean in $\mathbb{R}^d$ absolutely and has weak mean m_μ.*

Remark 3.11. Let $\mu \in ID$. Then the property (3.18) does not depend on the choice of a spherical decomposition of ν_μ. Indeed, if $(\bar{\nu}_\mu(dr), \lambda_r^\mu)$ and $(\bar{\nu}'(dr), \lambda_r'(d\xi))$ are two spherical decompositions of ν_μ, then we have

$$\int_{(1,\infty)} r\bar{\nu}'(dr) \left| \int_S \xi \lambda_r'(d\xi) \right| = \int_{(1,\infty)} r\bar{\nu}_\mu(dr) \left| \int_S \xi \lambda_r^\mu(d\xi) \right|,$$

since there is $c(r)$ satisfying (3.8)–(3.10) of Proposition 3.3. □

Remark 3.12. If $\mu \in ID$ has weak mean in $\mathbb{R}^d$ absolutely, then μ has weak mean in $\mathbb{R}^d$, since

$$\int_{1<|x|\le a} x\nu_\mu(dx) = \int_{(1,a]} r\bar{\nu}_\mu(dr) \int_S \xi \lambda_r^\mu(d\xi)$$

for $a > 1$. □

Remark 3.13. If $\mu \in ID$ has mean m_μ, then μ has weak mean m_μ absolutely, because finiteness of

$$\int_{|x|>1} |x| \nu_\mu(dx) = \int_{(1,\infty)} r\bar{\nu}_\mu(dr) \int_S \lambda_r^\mu(d\xi)$$

implies (3.18). □

Proposition 3.14. *Let $\mu \in ID$. If μ is symmetric (that is, $\mu(-B) = \mu(B)$ for all $B \in \mathscr{B}(\mathbb{R}^d)$), then μ has weak mean* 0 *absolutely.*

Proof. Assume that μ is symmetric. Then $C_\mu(z)$ is real. Hence ν_μ is symmetric and $\gamma_\mu = 0$. Thus $\int_{1<|x|\le a} x\nu_\mu(dx) = 0$ for $a > 1$. Hence it follows from Proposition 3.9 that μ has weak mean 0. Let $(\bar{\nu}_\mu(dr), \lambda_r^\mu)$ be a spherical decomposition of ν_μ. Then it follows from the symmetry of ν_μ that, for $\bar{\nu}_\mu$-a. e. r, λ_r^μ is symmetric, so that $\int_S \xi \lambda_r^\mu(d\xi) = 0$. Hence (3.18) holds. □

Proposition 3.15. *Let $\mu \in ID$ with ν_μ of polar product type. Assume that $\int_{\mathbb{R}^d} |x|\mu(dx) = \infty$. Then the following five conditions are equivalent.*
(a) *μ has weak mean in $\mathbb{R}^d$.*
(b) *μ has weak mean in $\mathbb{R}^d$ and $m_\mu^L = 0$.*
(c) *μ has weak mean in $\mathbb{R}^d$ absolutely.*
(d) *μ has weak mean in $\mathbb{R}^d$ absolutely and $m_\mu^L = 0$.*
(e) *λ_μ and $\bar{\nu}_\mu$ in* (3.11) *satisfy $\int_S \xi \lambda_\mu(d\xi) = 0$.*

Proof. Clearly (d) ⇒ (b) ⇒ (a) and (d) ⇒ (c) ⇒ (a). Since $\int_{1<|x|\le a} x\nu_\mu(dx) = \int_S \xi \lambda_\mu(d\xi) \int_{(1,a]} r\bar{\nu}_\mu(dr)$, and $\int_{(1,\infty)} r\nu_\mu(dr) = \infty$, (a) implies (e). Since $\bar{\nu}_\mu$ and λ_μ give a spherical decomposition of ν_μ, (e) implies (d). □

Example 3.16. Let μ be a 1-stable distribution on $\mathbb{R}^d$. Since $\int_S \xi \lambda_\mu(d\xi) = 0$ if and only if μ is strictly 1-stable, Proposition 3.15 gives equivalent characterizations of strict 1-stability. □

3.4 Stochastic integral mappings of infinitely divisible distributions

In this section let $f(s)$ be a locally square-integrable nonrandom function on $\mathbb{R}_+ = [0,\infty)$ (that is, $f(s)$ is extended real-valued, measurable, and $\int_0^t f(s)^2 ds < \infty$ for any $t \ge 0$). Then, it is known that, for any Lévy process $\{X_s^{(\rho)} : s \ge 0\}$ on $\mathbb{R}^d$, the stochastic integral $\int_E f(s) dX_s^{(\rho)}$ is definable for every bounded Borel set E on $\mathbb{R}_+$, and its law μ_E is infinitely divisible and satisfies

$$C_{\mu_E}(z) = \int_E C_\rho(f(s)z) ds, \quad z \in \mathbb{R}^d \tag{3.19}$$

with $\int_E |C_\rho(f(s)z)| ds < \infty$ (see [17, 31, 40, 41]). We write $\int_0^t f(s) dX_s^{(\rho)}$ for $\int_{[0,t]} f(s)$ $dX_s^{(\rho)}$.

Proposition 3.17. *Let* $\mu_t = \mathscr{L}\left(\int_0^t f(s)dX_s^{(\rho)}\right)$. *Then the triplet of* μ_t *is given by*

$$A_{\mu_t} = \int_0^t f(s)^2 A_\rho ds, \tag{3.20}$$

$$\nu_{\mu_t}(B) = \int_0^t ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu_\rho(dx), \quad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}), \tag{3.21}$$

$$\gamma_{\mu_t} = \int_0^t f(s)ds \left(\gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|f(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\nu_\rho(dx)\right), \tag{3.22}$$

$$\gamma_{\mu_t}^\sharp = \int_0^t f(s)ds \left(\gamma_\rho^\sharp + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu_\rho(dx)\right). \tag{3.23}$$

Proof. It follows from (1.1) and (3.19) with $E = [0,t]$ that

$$\begin{aligned} C_{\mu_t}(z) &= \int_0^t ds[\tfrac{-1}{2} f(s)^2 \langle z, A_\rho z\rangle + \int_{\mathbb{R}^d} (e^{i\langle f(s)z,x\rangle} - 1 - i\langle f(s)z, x\rangle 1_{\{|x|\le 1\}}(x)) \\ &\quad \nu_\rho(dx) + i f(s)\langle \gamma_\rho, z\rangle] \\ &= \frac{-1}{2}\int_0^t f(s)^2 ds \langle z, A_\rho z\rangle + \int_0^t ds \int_{\mathbb{R}^d} (e^{i\langle z, f(s)x\rangle} - 1 - i\langle z, f(s)x\rangle 1_{\{|f(s)x|\le 1\}}) \\ &\quad \nu_\rho(dx) + i\int_0^t ds \left(\int_{\mathbb{R}^d} \langle z, f(s)x\rangle (1_{\{|f(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\nu_\rho(dx) + f(s)\langle \gamma_\rho, z\rangle\right). \end{aligned}$$

Hence we have (3.20)–(3.22). Similarly we obtain (3.23) from (3.1). □

We say that the improper stochastic integral of f with respect to $X^{(\rho)}$ is *definable* if $\int_0^t f(s)dX_s^{(\rho)}$ is convergent in probability as $t \to \infty$. The limit is written as $\int_0^{\infty-} f(s)dX_s^{(\rho)}$; we define

$$\Phi_f \rho = \mathscr{L}\left(\int_0^{\infty-} f(s)dX_s^{(\rho)}\right). \tag{3.24}$$

The domain $\mathfrak{D}(\Phi_f)$ of Φ_f is defined by

$$\mathfrak{D}(\Phi_f) = \{\rho \in ID(\mathbb{R}^d)\colon \textstyle\int_0^{\infty-} f(s)dX_s^{(\rho)} \text{ is definable}\}.$$

We often say that $\Phi_f \rho$ is definable if $\int_0^{\infty-} f(s)dX_s^{(\rho)}$ is definable. It is known that $\Phi_f \rho$ is definable if and only if $\int_0^t C_\rho(f(s)z)ds$ is convergent as $t \to \infty$ for every $z \in \mathbb{R}^d$ (see [42]). If $\Phi_f \rho$ is definable, then

$$C_{\Phi_f \rho}(z) = \lim_{t\to\infty} \int_0^t C_\rho(f(s)z)ds, \quad z \in \mathbb{R}^d. \tag{3.25}$$

Three extended notions (essential, compensated, symmetrized) and one restricted notion of definability of improper stochastic integrals are introduced in [41, 42, 44]. Here we use the restricted notion and one extended notion. We say that $\int_0^{\infty-} f(s)dX_s^{(\rho)}$ is *absolutely definable* if

$$\int_0^\infty |C_\rho(f(s)z)|ds < \infty, \quad z \in \mathbb{R}^d. \tag{3.26}$$

Let

$$\mathfrak{D}^0(\Phi_f) = \{\rho \in ID(\mathbb{R}^d) \colon \int_0^{\infty-} f(s)dX_s^{(\rho)} \text{ is absolutely definable}\}.$$

We say that $\int_0^{\infty-} f(s)dX_s^{(\rho)}$ is *essentially definable* if, for some $\mathbb{R}^d$-valued function q_t on $\mathbb{R}_+$, $\int_0^t f(s)dX_s^{(\rho)} - q_t$ is convergent in probability as $t \to \infty$, which is equivalent to the property that $\int_0^t C_\rho(f(s)z)ds - i\langle q_t, z\rangle$ is convergent as $t \to \infty$ for every $z \in \mathbb{R}^d$. Let

$$\mathfrak{D}^{\mathrm{e}}(\Phi_f) = \{\rho \in ID(\mathbb{R}^d) \colon \int_0^{\infty-} f(s)dX_s^{(\rho)} \text{ is essentially definable}\}.$$

We have

$$\mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_f) \subset \mathfrak{D}^{\mathrm{e}}(\Phi_f). \tag{3.27}$$

Define the range $\mathfrak{R}(\Phi_f)$, the absolute range $\mathfrak{R}^0(\Phi_f)$, and the essential range $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$ of Φ_f as

$$\begin{aligned}
\mathfrak{R}(\Phi_f) &= \{\Phi_f\rho \colon \rho \in \mathfrak{D}(\Phi_f)\},\\
\mathfrak{R}^0(\Phi_f) &= \{\Phi_f\rho \colon \rho \in \mathfrak{D}^0(\Phi_f)\},\\
\mathfrak{R}^{\mathrm{e}}(\Phi_f) &= \{\mathscr{L}(\operatorname*{p-lim}_{t\to\infty}(\textstyle\int_0^t f(s)dX_s^{(\rho)} - q_t)) \colon \rho \in \mathfrak{D}^{\mathrm{e}}(\Phi_f) \text{ and}\\
&\qquad \operatorname*{p-lim}_{t\to\infty}(\textstyle\int_0^t f(s)dX_s^{(\rho)} - q_t) \text{ exists }\}.
\end{aligned}$$

Then

$$\mathfrak{R}^0(\Phi_f) \subset \mathfrak{R}(\Phi_f) \subset \mathfrak{R}^{\mathrm{e}}(\Phi_f). \tag{3.28}$$

We will use the following fact.

Proposition 3.18. *Let $\rho \in ID(\mathbb{R}^d)$.*

(i) $\rho \in \mathfrak{D}(\Phi_f)$ *if and only if the following three conditions are satisfied.*

$$\int_0^\infty f(s)^2(\mathrm{tr}A_\rho)ds < \infty, \tag{3.29}$$

$$\int_0^\infty ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu_\rho(dx) < \infty, \tag{3.30}$$

$$\gamma_{\mu_t} \text{ of (3.22) is convergent in } \mathbb{R}^d \text{ as } t \to \infty. \tag{3.31}$$

(3.31) *can be replaced by*

$$\gamma^\sharp_{\mu_t} \text{ of (3.23) is convergent in } \mathbb{R}^d \text{ as } t \to \infty. \tag{3.32}$$

(ii) *If $\rho \in \mathfrak{D}(\Phi_f)$, then the triplet of $\mu = \Phi_f\rho$ is given by*

$$A_\mu = \int_0^\infty f(s)^2 A_\rho ds, \tag{3.33}$$

$$\nu_\mu(B)=\int_0^\infty ds\int_{\mathbb{R}^d}1_B(f(s)x)\nu_\rho(dx),\quad B\in\mathscr{B}(\mathbb{R}^d\setminus\{0\}),\tag{3.34}$$

$$\gamma_\mu=\lim_{t\to\infty}\gamma_{\mu_t},\tag{3.35}$$

$$\gamma_\mu^\sharp=\lim_{t\to\infty}\gamma_{\mu_t}^\sharp.\tag{3.36}$$

(iii) $\rho\in\mathfrak{D}^0(\Phi_f)$ *if and only if* (3.29), (3.30), *and*

$$\int_0^\infty|f(s)|ds\left|\gamma_\rho+\int_{\mathbb{R}^d}x(1_{\{|f(s)x|\le1\}}-1_{\{|x|\le1\}})\nu_\rho(dx)\right|<\infty,\tag{3.37}$$

or, equivalently, (3.29), (3.30), *and*

$$\int_0^\infty|f(s)|ds\left|\gamma_\rho^\sharp+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu_\rho(dx)\right|<\infty.\tag{3.38}$$

(iv) $\rho\in\mathfrak{D}^{\mathrm{e}}(\Phi_f)$ *if and only if* (3.29) *and* (3.30) *are satisfied.*

Proof. If we ignore the statements related to γ_ρ, γ_{μ_t}, and γ_μ and retain those related to $\gamma_\rho^\sharp$, $\gamma_{\mu_t}^\sharp$, and $\gamma_\mu^\sharp$, this proposition is proved in Lemma 5.4 and Propositions 5.5–5.6 of [41], Proposition 2.6 of [42], and Propositions 2.1–2.3 of [43]. Let us give remarks concerning the statements related to γ_ρ, γ_{μ_t}, and γ_μ. An important point is that the convergence of μ_t as $t\to\infty$ is not only convergence of infinitely divisible distributions but also with $\langle z,A_{\mu_t}z\rangle$, $z\in\mathbb{R}^d$, and ν_{μ_t} increasing with t. If (3.30), (3.32), (3.34), and (3.36) hold, then

$$\int_{|x|\le1}\frac{x|x|^2}{1+|x|^2}\nu_{\mu_t}(dx)\to\int_{|x|\le1}\frac{x|x|^2}{1+|x|^2}\nu_\mu(dx),$$
$$\int_{|x|>1}\frac{x}{1+|x|^2}\nu_{\mu_t}(dx)\to\int_{|x|>1}\frac{x}{1+|x|^2}\nu_\mu(dx),\quad t\to\infty,$$

by virtue of the increase of ν_{μ_t} in t, and hence, from (3.2), γ_{μ_t} is convergent to γ_μ, that is, (3.31) and (3.35) hold. Conversely, (3.30), (3.31), (3.34), and (3.35) together imply (3.32) and (3.36). The proof of the equivalence of $\rho\in\mathfrak{D}^0(\Phi_f)$ to (3.29), (3.30), and (3.37) is as follows, which is similar to the proof of Proposition 2.3 of [43].

For $u\in\mathbb{R}$ define $\rho_u(B)=\int_{\mathbb{R}^d}1_B(ux)\rho(dx)$, $B\in\mathscr{B}(\mathbb{R}^d)$. Then $\rho_u\in ID$, $A_{\rho_u}=u^2A_\rho$, $\nu_{\rho_u}(B)=\int_{\mathbb{R}^d}1_B(ux)\nu_\rho(dx)$ for $B\in\mathscr{B}(\mathbb{R}^d\setminus\{0\})$, and $\gamma_{\rho_u}=u\gamma_\rho+\int_{\mathbb{R}^d}ux(1_{\{|ux|\le1\}}-1_{\{|x|\le1\}})\nu_\rho(dx)$. We have

$$|C_\rho(uz)|=|C_{\rho_u}(z)|\le\frac{|z|^2}{2}\mathrm{tr}A_{\rho_u}+\left(\frac{|z|^2}{2}+2\right)\int_{\mathbb{R}^d}(|x|^2\wedge1)\nu_{\rho_u}(dx)+|\gamma_{\rho_u}|\,|z|.$$

If (3.29), (3.30), and (3.37) hold, then $\int_0^\infty(\mathrm{tr}A_{\rho_{f(s)}})ds<\infty$, $\int_0^\infty ds\int_{\mathbb{R}^d}(|x|^2\wedge1)\nu_{\rho_{f(s)}}(dx)<\infty$, and $\int_0^\infty|\gamma_{\rho_{f(s)}}|ds<\infty$ and hence $\int_0^\infty|C_\rho(f(s)z)|ds<\infty$, that is, $\rho\in\mathfrak{D}^0(\Phi_f)$.

Conversely, assume that $\rho \in \mathfrak{D}^0(\Phi_f)$. Then (3.29) and (3.30) hold, since $\rho \in \mathfrak{D}(\Phi_f)$. Notice that $\operatorname{Im} C_\rho(f(s)z) = I_1(s) + I_2(s)$, where

$$I_1(s) = \int_{\mathbb{R}^d} (\sin\langle z, f(s)x\rangle - \langle z, f(s)x\rangle 1_{\{|f(s)x|\le 1\}})\nu_\rho(dx),$$
$$I_2(s) = \int_{\mathbb{R}^d} \langle z, f(s)x\rangle (1_{\{|f(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\nu_\rho(dx) + \langle f(s)\gamma_\rho, z\rangle = \langle \gamma_{\rho_{f(s)}}, z\rangle.$$

Since
$$|I_1(s)| \le \left(\frac{|z|^3}{6} + 1\right) \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_{\rho_{f(s)}}(dx),$$
we have $\int_0^\infty |I_1(s)|ds < \infty$ from (3.30). Hence it follows from

$$\infty > \int_0^\infty |C_\rho(f(s)z)|ds \ge \int_0^\infty |\operatorname{Im} C_\rho(f(s)z)|ds$$

that $\int_0^\infty |\langle \gamma_{\rho_{f(s)}}, z\rangle| ds < \infty$ for any $z \in \mathbb{R}^d$. Therefore $\int_0^\infty |\gamma_{\rho_{f(s)}}| ds < \infty$, which is equivalent to (3.37). □

The following two propositions state the "linearity" of Φ_f, but we will not use them except in Section 5.2. For $\rho \in ID(\mathbb{R}^d)$ and $t \ge 0$, the distinguished tth power of $\widehat{\rho}(z)$ in the terminology of [39] is denoted by $\widehat{\rho}(z)^t$ (that is, $\widehat{\rho}(z)^t = \exp(tC_\rho(z))$) and the infinitely divisible distribution with characteristic function $\widehat{\rho}(z)^t$ is denoted by ρ^t.

Proposition 3.19. *Let $t \ge 0$. If $\rho \in \mathfrak{D}(\Phi_f)$, then $\rho^t \in \mathfrak{D}(\Phi_f)$ and $\Phi_f(\rho^t) = (\Phi_f\rho)^t$. If $\rho \in \mathfrak{D}^0(\Phi_f)$ [resp. $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$], then $\rho^t \in \mathfrak{D}^0(\Phi_f)$ [resp. $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$].*

Proof. Use the relation $C_{\rho^t}(z) = tC_\rho(z)$. □

Proposition 3.20. *If $\rho, \rho' \in \mathfrak{D}(\Phi_f)$, then $\rho * \rho' \in \mathfrak{D}(\Phi_f)$ and $\Phi_f(\rho * \rho') = (\Phi_f\rho) * (\Phi_f\rho')$. If $\rho, \rho' \in \mathfrak{D}^0(\Phi_f)$ [resp. $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$], then $\rho * \rho' \in \mathfrak{D}^0(\Phi_f)$ [resp. $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$].*

Proof. Use the relation $C_{\rho*\rho'}(z) = C_\rho(z) + C_{\rho'}(z)$. □

Remark 3.21. Sato and Yamazato [45] proves the continuity of the mapping Φ in some sense. Results in Rajput and Rosinski [31] suggest continuity in some sense of the restriction of Φ_f to $\mathfrak{D}^0(\Phi_f)$. □

Remark 3.22. If $|f_2| \le |f_1|$, then $\mathfrak{D}^0(\Phi_{f_1}) \subset \mathfrak{D}^0(\Phi_{f_2})$. We express this fact by saying that the class $\mathfrak{D}^0(\Phi_f)$ is monotonic with respect to f. In this terminology, $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$ is also monotonic with respect to f, but $\mathfrak{D}(\Phi_f)$ is not monotonic with respect to f. The latter fact is related to some properties of martingale Lévy processes. See Sato [43]. □

Remark 3.23. Up to this point in this section we have assumed that f is a locally square-integrable function on $\mathbb{R}_+$. Given $\rho \in ID$, $\int_E f(s)dX_s^{(\rho)}$ can possibly be defined for all bounded Borel sets E on $\mathbb{R}_+$ for a function f satisfying a weaker condition. However, we can prove that if f is a measurable nonrandom function on $\mathbb{R}_+$

such that $\int_E f(s)dX_s^{(\rho)}$ is defined for all bounded Borel sets E on $\mathbb{R}_+$ and for all $\rho \in ID$, then f is locally square-integrable on $\mathbb{R}_+$ ([41]). □

Remark 3.24. General treatment (with random integrands in general) of improper stochastic integrals and stochastic integrals up to infinity from the semimartingale point of view is made by Cherny and Shiryaev [6]. Stochastic integrals of nonrandom functions with respect to an infinitely divisible random measure $\Lambda(B)$ for B in a σ-ring of subsets of a general parameter space are studied by Rajput and Rosinski [31]. The integrability condition suggests that our absolutely definable improper stochastic integral of a nonrandom function with respect to a Lévy process should be identical with the stochastic integral up to infinity of Cherny and Shiryaev [6] and with the stochastic integral of a Λ-integrable function of Rajput and Rosinski [31]. In our set-up, improper stochastic integrals in more general cases are studied in [41, 44]. □

3.5 Transformation of Lévy measures

Let $f(s)$ be a locally square-integrable nonrandom function on $\mathbb{R}_+$. Suggested by the equation (3.34), we introduce the transformation Φ_f^L in the following way.

Definition 3.25. *For $\nu \in \mathfrak{M}^L = \mathfrak{M}^L(\mathbb{R}^d)$, let $\widetilde{\nu}$ be a measure on $\mathbb{R}^d$ defined by $\widetilde{\nu}(\{0\}) = 0$ and*

$$\widetilde{\nu}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx), \quad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}). \tag{3.39}$$

Define

$$\mathfrak{D}(\Phi_f^L) = \{\nu \in \mathfrak{M}^L : \widetilde{\nu} \in \mathfrak{M}^L\}$$

and $\Phi_f^L \nu = \widetilde{\nu}$ for $\nu \in \mathfrak{D}(\Phi_f^L)$. The range is defined by

$$\mathfrak{R}(\Phi_f^L) = \{\Phi_f^L \nu : \nu \in \mathfrak{D}(\Phi_f^L)\}.$$

Remark 3.26. Suppose that $\int_0^\infty |f(s)|ds > 0$. For a measure ν on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$, define a measure $\widetilde{\nu}$ on $\mathbb{R}^d$ by $\widetilde{\nu}(\{0\}) = 0$ and by (3.39). If $\widetilde{\nu} \in \mathfrak{M}^L$, then $\nu \in \mathfrak{M}^L$. Indeed, choose $0 < a \le 1$ and $E \in \mathscr{B}(\mathbb{R}_+^\circ)$ with Lebesgue measure b such that $|f(s)| \ge a$ for $s \in E$, and observe that

$$\begin{aligned}\int_{R^d} (|x|^2 \wedge 1)\widetilde{\nu}(dx) &= \int_0^\infty ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu(dx) \ge \int_E ds \int_{\mathbb{R}^d} ((a^2|x|^2) \wedge 1)\nu(dx) \\ &\ge ba^2 \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx),\end{aligned}$$

where E can be chosen with b finite and positive. □

The essential range $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$ is connected with $\mathfrak{R}(\Phi_f^L)$.

Proposition 3.27. *Suppose that* $0 < \int_0^\infty f(s)^2 ds < \infty$. *Then* $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$ *is the class of* $\mu \in ID$ *with Lévy measure* ν_μ *being in* $\mathfrak{R}(\Phi_f^L)$.

Proof. If $\mu \in \mathfrak{R}^{\mathrm{e}}(\Phi_f)$, then we have $\nu_\mu \in \mathfrak{R}(\Phi_f^L)$ immediately from Proposition 3.18 (iv) and Definition 3.25.

Conversely, let $\mu \in ID$ such that $\nu_\mu = \Phi_f^L \nu$ for some $\nu \in \mathfrak{D}(\Phi_f^L)$. Let $A = (\int_0^\infty f(s)^2 ds)^{-1} A_\mu$ and let $\rho \in ID$ be such that $(A_\rho, \nu_\rho, \gamma_\rho) = (A, \nu, 0)$. Let

$$q_t = \int_0^t f(s)ds \int_{\mathbb{R}^d} x(1_{\{|f(s)x|\leq 1\}} - 1_{\{|x|\leq 1\}})\nu_\rho(dx) - \gamma_\mu.$$

Then $\mathscr{L}(\int_0^t f(s)dX_s^{(\rho)} - q_t)$ has triplet $(\widetilde{A}_t, \widetilde{\nu}_t, \widetilde{\gamma}_t)$, where $\widetilde{A}_t$ and $\widetilde{\nu}_t$ are given by the right-hand side of (3.20) and (3.21), and $\widetilde{\gamma}_t = \gamma_\mu$. Using Lemma 5.4 of [41] and the argument in the proof of Proposition 3.18, we see that $\int_0^t f(s)dX_s^{(\mu)} - q_t$ converges in probability as $t \to \infty$. The limit distribution equals μ. Hence $\mu \in \mathfrak{R}^{\mathrm{e}}(\Phi_f)$. □

4 First two-parameter extension $K_{p,\alpha}$ of the class L of selfdecomposable distributions

4.1 Φ_f and Φ_f^L for $f = \varphi_\alpha$

We give some consequences of the conditions

$$\text{there are } a_1, a_2 > 0 \text{ such that } e^{-a_2 s} \leq \varphi_0(s) \leq e^{-a_1 s} \text{ for all large } s, \tag{4.1}$$

and

$$\varphi_\alpha(s) \asymp s^{-1/\alpha} \quad \text{with } \alpha \in (0,\infty) \text{ as } s \to \infty. \tag{4.2}$$

In general, for two functions f and g we write $f(s) \asymp g(s)$, $s \to \infty$, if there are positive constants a_1 and a_2 such that $0 < a_1 g(s) \leqslant f(s) \leqslant a_2 g(s)$ for all large s. The following description of the domains is known.

Theorem 4.1. *Let* $0 \leq \alpha < \infty$. *Suppose that* φ_α *is locally square-integrable on* $\mathbb{R}_+$ *and satisfies* (4.1)–(4.2).

(i) *If* $\alpha = 0$, *then*

$$\mathfrak{D}(\Phi_{\varphi_0}^L) = \{\nu \in \mathfrak{M}^L \colon \int_{|x|>1} \log|x| \nu(dx) < \infty\}.$$

(ii) *If* $0 < \alpha < 2$, *then*

$$\mathfrak{D}(\Phi_{\varphi_\alpha}^L) = \{\nu \in \mathfrak{M}^L \colon \int_{|x|>1} |x|^\alpha \nu(dx) < \infty\}.$$

(iii) *If* $\alpha \geq 2$, *then* $\mathfrak{D}(\Phi_{\varphi_\alpha}^L) = \{\delta_0\}$.

Proof is given in the same way as that of Lemma 2.7 of [42] and Proposition 4.3 of [43]. A similar proof will be given to Theorem 6.2.

Theorem 4.2. *Let $0 \le \alpha < \infty$. Suppose that φ_α is locally square-integrable on $\mathbb{R}_+$ and satisfies* (4.1)–(4.2).

(i) *If $\alpha = 0$, then*

$$\mathfrak{D}^0(\Phi_{\varphi_0}) = \mathfrak{D}(\Phi_{\varphi_0}) = \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_0}) = \{\rho \in ID\colon \textstyle\int_{|x|>1} \log|x|\,\nu_\rho(dx) < \infty\}. \tag{4.3}$$

(ii) *If $0 < \alpha < 1$, then*

$$\mathfrak{D}^0(\Phi_{\varphi_\alpha}) = \mathfrak{D}(\Phi_{\varphi_\alpha}) = \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_\alpha}) = \{\rho \in ID\colon \textstyle\int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty\}. \tag{4.4}$$

(iii) *If $\alpha = 1$, then*

$$\mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_1}) = \{\rho \in ID\colon \textstyle\int_{|x|>1} |x|\nu_\rho(dx) < \infty\} \tag{4.5}$$

$$\begin{aligned}\mathfrak{D}(\Phi_{\varphi_1}) = \{\rho \in ID\colon & \textstyle\int_{|x|>1} |x|\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\rho(dx) = 0,\\ & \lim_{t\to\infty} \int_0^t ds \int_{|\varphi_1(s)x|>1} \varphi_1(s)x\,\nu_\rho(dx) \text{ exists in } \mathbb{R}^d\}\\ = \{\rho \in ID\colon & \textstyle\int_{|x|>1} |x|\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\rho(dx) = 0,\\ & \lim_{t\to\infty} \int_0^t ds \int_{\mathbb{R}^d} \frac{\varphi_1(s)x\,|\varphi_1(s)x|^2}{1+|\varphi_1(s)x|^2} \nu_\rho(dx) \text{ exists in } \mathbb{R}^d\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathfrak{D}^0(\Phi_{\varphi_1}) = \{\rho \in ID\colon & \textstyle\int_{|x|>1} |x|\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\rho(dx) = 0,\\ & \int_0^\infty ds \left| \int_{|\varphi_1(s)x|>1} \varphi_1(s)x\,\nu_\rho(dx)\right| < \infty\}\\ = \{\rho \in ID\colon & \textstyle\int_{|x|>1} |x|\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\rho(dx) = 0,\\ & \int_0^\infty ds \left| \int_{\mathbb{R}^d} \frac{\varphi_1(s)x\,|\varphi_1(s)x|^2}{1+|\varphi_1(s)x|^2} \nu_\rho(dx)\right| < \infty\}.\end{aligned} \tag{4.7}$$

(iv) *If $1 < \alpha < 2$, then*

$$\mathfrak{D}^0(\Phi_{\varphi_\alpha}) = \mathfrak{D}(\Phi_{\varphi_\alpha}) \subsetneqq \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_\alpha}), \tag{4.8}$$

$$\mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_\alpha}) = \{\rho \in ID\colon \textstyle\int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty\}, \tag{4.9}$$

$$\mathfrak{D}^0(\Phi_{\varphi_\alpha}) = \mathfrak{D}(\Phi_{\varphi_\alpha}) = \{\mu \in \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_\alpha})\colon \textstyle\int_{\mathbb{R}^d} x\rho(dx) = 0\}. \tag{4.10}$$

(v) *If $\alpha \ge 2$, then*

$$\mathfrak{D}^0(\Phi_{\varphi_\alpha}) = \mathfrak{D}(\Phi_{\varphi_\alpha}) = \{\delta_0\} \subsetneqq \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_\alpha}) = \{\delta_\gamma\colon \gamma \in \mathbb{R}^d\}. \tag{4.11}$$

Recall the following. If $\rho \in ID$, then $\int_{|x|>1} \log|x|\nu_\rho(dx) < \infty$ and $\int_{|x|>1} \log|x|\,\rho(dx) < \infty$ are equivalent. If $\rho \in ID$ and $\alpha > 0$, then $\int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty$ and $\int_{\mathbb{R}^d} |x|^\alpha \rho(dx) < \infty$ are equivalent. See Theorem 25.3 of [39].

Lemma 4.3. *Let $\rho \in ID$ and $\int_{\mathbb{R}^d} |x|\rho(dx) < \infty$. Then*

$$\begin{aligned}\int_{\mathbb{R}^d} x\rho(dx) = 0 \quad &\Leftrightarrow \quad \gamma_\rho = -\int_{|x|>1} x\,\nu_\rho(dx) \\ &\Leftrightarrow \quad \gamma_\rho^\sharp = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\nu_\rho(dx).\end{aligned} \tag{4.12}$$

Proof. Straightforward from (3.2) and (3.17). □

Proof of Theorem 4.2. Except assertion (iii), these are shown in Theorem 2.4 of [42]. A similar proof will be given to Theorem 6.3. Let us prove (iii). For the description of $\mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_1})$, combine Proposition 3.18 (iv) with Theorem 4.1. In order to prove (4.6) for $\mathfrak{D}(\Phi_{\varphi_1})$, first note that $\rho \in \mathfrak{D}(\Phi_{\varphi_1})$ if and only if $\rho \in \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_1})$ and (3.31) holds (Proposition 3.18 (i)). Recall that $\varphi_1(s) \asymp s^{-1}$ as $s \to \infty$. Assume that $\rho \in \mathfrak{D}(\Phi_{\varphi_1})$. Then $\int_{|x|>1} |x|\nu_\rho(dx) < \infty$. We have $1_{\{|\varphi_1(s)x|\le 1\}} - 1_{\{|x|\le 1\}} \to 1_{\{|x|>1\}}$ as $s \to \infty$. It follows from the dominated convergence theorem that

$$\int_{\mathbb{R}^d} x(1_{\{|\varphi_1(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\nu_\rho(dx) \to \int_{|x|>1} x\,\nu_\rho(dx), \quad s \to \infty,$$

since $|x(1_{\{|\varphi_1(s)x|\le 1\}} - 1_{\{|x|\le 1\}})|$ is bounded by $|x|$ for all x and equals 0 for $|x| \le 1$ and $|\varphi_1(s)| \le 1$. Since γ_{μ_t} of (3.22) is convergent, it follows that γ_ρ satisfies the condition in (4.12). Hence $\int_{\mathbb{R}^d} x\rho(dx) = 0$. Replacing γ_ρ in (3.22) by that of (4.12), we obtain

$$\gamma_{\mu_t} = -\int_0^t ds \int_{|\varphi_1(s)x|>1} \varphi_1(s)x\,\nu_\rho(dx).$$

Hence μ belongs to the right-hand side of the first equality of (4.6). The converse direction is similar. This proves the first equality of (4.6). The proof of the second equality of (4.6) is done in the same idea. In order to show (4.7) for $\mathfrak{D}^0(\Phi_{\varphi_1})$, notice that $\rho \in \mathfrak{D}^0(\Phi_{\varphi_1})$ if and only if $\rho \in \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_1})$ and (3.37) holds (Proposition 3.18 (iii)) and use Lemma 4.3. □

Finer results are given in the case $\alpha = 1$.

Theorem 4.4. *Suppose that φ_1 is locally square-integrable on $\mathbb{R}_+$ and satisfies $\varphi_1(s) \asymp s^{-1}$ as $s \to \infty$. Suppose, in addition, that*

$$\int_1^\infty |\varphi_1(s) - cs^{-1}|ds < \infty \tag{4.13}$$

with some $c > 0$. Then

$$\mathfrak{D}^0(\Phi_{\varphi_1}) \subsetneqq \mathfrak{D}(\Phi_{\varphi_1}) \subsetneqq \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_1}), \tag{4.14}$$

$$\begin{aligned}\mathfrak{D}(\Phi_{\varphi_1}) = \{\rho \in ID\colon\ &\textstyle\int_{|x|>1}|x|\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\rho(dx) = 0, \\ &\lim_{t\to\infty}\int_1^t s^{-1}ds\int_{|x|>s} x\nu_\rho(dx) \text{ exists in } \mathbb{R}^d\},\end{aligned} \tag{4.15}$$

$$\mathfrak{D}^0(\Phi_{\varphi_1}) = \{\rho \in ID\colon \textstyle\int_{|x|>1} |x| \nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\rho(dx) = 0,$$
$$\int_1^\infty s^{-1} ds \left| \int_{|x|>s} x\nu_\rho(dx) \right| < \infty\}. \tag{4.16}$$

This is Theorem 2.8 of [42].

For $\alpha < 1$ description of the ranges is simple.

Proposition 4.5. *Let* $0 \le \alpha < 1$. *Suppose that* φ_α *is locally square-integrable on* $\mathbb{R}_+$ *and satisfies* (4.1)–(4.2). *Suppose further that* $\varphi_\alpha \ge 0$. *Then*

$$\mathfrak{R}^0(\Phi_{\varphi_\alpha}) = \mathfrak{R}(\Phi_{\varphi_\alpha}) = \mathfrak{R}^{\mathrm{e}}(\Phi_{\varphi_\alpha}) = \{\mu \in ID\colon \nu_\mu \in \mathfrak{R}(\Phi^L_{\varphi_\alpha})\}. \tag{4.17}$$

Proof. It is known in (3.28) that $\mathfrak{R}^0(\Phi_{\varphi_\alpha}) \subset \mathfrak{R}(\Phi_{\varphi_\alpha}) \subset \mathfrak{R}^{\mathrm{e}}(\Phi_{\varphi_\alpha})$. Suppose that $\mu \in \mathfrak{R}^{\mathrm{e}}(\Phi_{\varphi_\alpha})$. Then μ is the law of $\operatorname*{p\text{-}lim}_{t\to\infty}(\int_0^t \varphi_\alpha(s) dX_s^{(\rho)} - q_t))$ for some $\rho \in \mathfrak{D}^{\mathrm{e}}(\Phi_{\varphi_\alpha})$ and some q_t. Since $\alpha < 1$, it follows from Theorem 4.2 that $\rho \in \mathfrak{D}^0(\Phi_{\varphi_\alpha})$. Hence $\int_0^t \varphi_\alpha(s) dX_s^{(\rho)}$ is convergent in probability as $t \to \infty$. Thus q_t tends to some $q \in \mathbb{R}^d$. It follows that $\mu = \widetilde{\mu} * \delta_{-q}$ for some $\widetilde{\mu} \in \mathfrak{R}^0(\Phi_{\varphi_\alpha})$. Since $0 < \int_0^\infty \varphi_\alpha(s) ds < \infty$, we see that μ itself belongs to $\mathfrak{R}^0(\Phi_{\varphi_\alpha})$. The assertion $\mathfrak{R}^{\mathrm{e}}(\Phi_{\varphi_\alpha}) = \{\mu \in ID\colon \nu_\mu \in \mathfrak{R}(\Phi^L_{\varphi_\alpha})\}$ comes from Proposition 3.27. □

4.2 $\bar{\Phi}_{p,\alpha}$ and $\bar{\Phi}^L_{p,\alpha}$

Let $-\infty < \alpha < \infty$ and $0 < p \le \infty$. In Section 1.6 we have introduced the two-parameter family of mappings $\bar{\Phi}_{p,\alpha}$. Namely, starting from the function $s = \bar{g}_{p,\alpha}(t)$ of (1.13), we define its inverse function $t = \bar{f}_{p,\alpha}(s)$ for $0 \le s < \bar{a}_{p,\alpha} = \bar{g}_{p,\alpha}(0+)$, where $\bar{g}_{p,\alpha}(0+) = \Gamma_{-\alpha}/\Gamma_{p-\alpha}$ for $\alpha < 0$ and ∞ for $\alpha \ge 0$; if $\alpha < 0$, then $\bar{f}_{p,\alpha}(s)$ is defined to be zero for $s \ge \bar{a}_{p,\alpha}$; then we define

$$\bar{\Phi}_{p,\alpha}\rho = \mathscr{L}\left(\int_0^{\infty-} \bar{f}_{p,\alpha}(s) dX_s^{(\rho)}\right) \tag{4.18}$$

with $\mathfrak{D}(\bar{\Phi}_{p,\alpha})$ being the class of $\rho \in ID$ such that the improper stochastic integral in (4.18) is definable.

Note the following special cases. If $p = 1$ and $\alpha = 0$, then

$$\bar{g}_{1,0}(t) = -\log t, \quad 0 < t \le 1, \qquad \bar{f}_{1,0}(s) = e^{-s}, \quad s \ge 0. \tag{4.19}$$

Thus $\bar{\Phi}_{1,0} = \Phi$, where Φ is given by (1.11). If $p = 1$ and $\alpha \ne 0$, then

$$\bar{g}_{1,\alpha}(t) = \begin{cases} \alpha^{-1}(t^{-\alpha} - 1), & 0 < t \le 1 \qquad \text{for } \alpha > 0, \\ (-\alpha)^{-1}(1 - t^{-\alpha}), & 0 \le t \le 1 \qquad \text{for } \alpha < 0, \end{cases} \tag{4.20}$$

$$\bar{f}_{1,\alpha}(s) = \begin{cases} (1+\alpha s)^{-1/\alpha}, \quad s \geq 0 & \text{for } \alpha > 0, \\ (1-(-\alpha)s)^{1/(-\alpha)}, \quad 0 \leq s \leq (-\alpha)^{-1} & \text{for } \alpha < 0. \end{cases} \tag{4.21}$$

If $p > 0$ and $\alpha = -1$, then

$$\bar{g}_{p,-1}(t) = c_{p+1}(1-t)^p, \quad 0 \leq t \leq 1, \tag{4.22}$$

$$\bar{f}_{p,-1}(s) = 1 - (\Gamma_{p+1}s)^{1/p}, \quad 0 \leq s \leq c_{p+1}. \tag{4.23}$$

Asymptotic behaviors of $\bar{f}_{p,\alpha}(s)$ for $\alpha \geq 0$ are as follows. This is given in Proposition 1.1 of [42] without proof.

Proposition 4.6. *As $s \to \infty$,*

$$\bar{f}_{p,0}(s) \sim \exp(c - \Gamma_p s) \text{ for } p > 0, \tag{4.24}$$

$$\bar{f}_{p,\alpha}(s) \sim (\alpha\Gamma_p s)^{-1/\alpha} \text{ for } \alpha > 0 \text{ and } p > 0, \tag{4.25}$$

$$\bar{f}_{p,1}(s) = (\Gamma_p s)^{-1} - (1-p)(\Gamma_p s)^{-2}\log s + O(s^{-2}) \text{ for } p > 0, \tag{4.26}$$

where

$$c = (p-1)\int_0^1 (1-u)^{p-2}\log u\,du. \tag{4.27}$$

Proof. (4.24): As $t \downarrow 0$,

$$\begin{aligned} \bar{g}_{p,0}(t) &= c_p \int_t^1 u^{-1}du + c_p \int_t^1 ((1-u)^{p-1}-1)u^{-1}du \\ &= -c_p \log t + c_p \int_0^1 ((1-u)^{p-1}-1)u^{-1}du + o(1) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 ((1-u)^{p-1}-1)u^{-1}du &= (1-p)\int_0^1 u^{-1}du \int_0^u (1-v)^{p-2}dv \\ &= (1-p)\int_0^1 (1-v)^{p-2}\log\frac{1}{v}dv = c. \end{aligned}$$

It follows that

$$s = c_p(-\log \bar{f}_{p,0}(s) + c) + o(1), \quad s \to \infty,$$

that is, (4.24) holds.

(4.25): Let $\alpha > 0$ and $p > 0$. As $t \downarrow 0$,

$$\begin{aligned} \bar{g}_{p,\alpha}(t) &= c_p \int_t^1 u^{-\alpha-1}du + c_p \int_t^1 ((1-u)^{p-1}-1)u^{-\alpha-1}du \\ &= \alpha^{-1}c_p t^{-\alpha} + O(t^{-\alpha+1}). \end{aligned}$$

Hence

$$s = \bar{f}_{p,\alpha}(s)^{-\alpha}(\alpha^{-1}c_p + o(1)), \quad s \to \infty.$$

(4.26): Let $p>0$. We have

$$\bar{g}_{p,1}(t)=c_p t^{-1}-(1-p)c_p\log t+O(1),\quad t\downarrow 0,$$

since

$$\begin{aligned}\bar{g}_{p,1}(t)&=c_p\left(t^{-1}+\int_t^1((1-u)^{p-1}-1)u^{-2}du-1\right)\\&=c_p\left(t^{-1}+(1-p)\int_t^1 udu+\int_t^1((1-u)^{p-1}-1-(1-p)u)u^{-2}du-1\right).\end{aligned}$$

Hence

$$s=c_p\bar{f}_{p,1}(s)^{-1}-(1-p)c_p\log\bar{f}_{p,1}(s)+O(1),\quad s\to\infty,$$

that is,

$$\bar{f}_{p,1}(s)=c_p s^{-1}-(1-p)c_p s^{-1}\bar{f}_{p,1}(s)\log\bar{f}_{p,1}(s)+O(s^{-1}\bar{f}_{p,1}(s)).\tag{4.28}$$

On the other hand we have

$$\begin{gathered}\bar{g}_{p,1}(t)=c_p t^{-1}+o(t^{-1}),\quad t\downarrow 0,\\ s=c_p\bar{f}_{p,1}(s)^{-1}+o(\bar{f}_{p,1}(s)^{-1}),\quad s\to\infty.\\ \bar{f}_{p,1}(s)=c_p s^{-1}(1+o(1)),\end{gathered}$$

successively. The last formula and (4.28) yield (4.26) with $o(s^{-2}\log s)$ in place of $O(s^{-2})$. Then this and (4.28) give (4.26). □

If $\alpha<0$, then $\mathfrak{D}^0(\bar{\Phi}_{p,\alpha})=\mathfrak{D}(\bar{\Phi}_{p,\alpha})=\mathfrak{D}^{\mathrm{e}}(\bar{\Phi}_{p,\alpha})=ID(\mathbb{R}^d)$. If $\alpha\geq 0$, then $\mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$, $\mathfrak{D}(\bar{\Phi}_{p,\alpha})$, and $\mathfrak{D}^{\mathrm{e}}(\bar{\Phi}_{p,\alpha})$ are described by Theorems 4.2 and 4.4 by virtue of Proposition 4.6. As a consequence, they do not depend on p. We notice that $\bar{\Phi}_{p,\alpha}$ is trivial if $\alpha\geq 2$.

For $-\infty<\alpha<\infty$ and $p>0$ we define $\bar{\Phi}^L_{p,\alpha}=\Phi^L_f$ with $f=\bar{f}_{p,\alpha}$ as in Definition 3.25. Again by Proposition 4.6, Theorem 4.1 is applied to the description of $\mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$, which does not depend on p. If $\alpha\geq 2$, then $\bar{\Phi}^L_{p,\alpha}$ is trivial. If $\alpha<0$, then $\mathfrak{D}(\bar{\Phi}^L_{p,\alpha})=\mathfrak{M}^L$.

If $\nu\in\mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$, then

$$\begin{aligned}\bar{\Phi}^L_{p,\alpha}\nu(B)&=\int_0^\infty ds\int_{\mathbb{R}^d}1_B(\bar{f}_{p,\alpha}(s)x)\nu(dx)\\&=-\int_0^1 d\bar{g}_{p,\alpha}(t)\int_{\mathbb{R}^d}1_B(tx)\nu(dx)\\&=c_p\int_0^1(1-t)^{p-1}t^{-\alpha-1}dt\int_{\mathbb{R}^d}1_B(tx)\nu(dx)\end{aligned}\tag{4.29}$$

for $B\in\mathscr{B}(\mathbb{R}^d\setminus\{0\})$. This shows that $\bar{\Phi}^L_{p,\alpha}$ is an example of Υ-transformations studied by Barndorff-Nielsen, Rosiński, and Thorbjørnsen [2].

The family $\bar{\Phi}^L_{p,\alpha}$ satisfies the following identity.

Theorem 4.7. *Let $-\infty < \alpha < 2$, $p > 0$, and $q > 0$. Then*

$$\bar{\Phi}^L_{p+q,\alpha} = \bar{\Phi}^L_{q,\alpha-p}\bar{\Phi}^L_{p,\alpha} = \bar{\Phi}^L_{p,\alpha}\bar{\Phi}^L_{q,\alpha-p}. \tag{4.30}$$

Proof. Let $\nu \in \mathfrak{M}^L(\mathbb{R}^d)$. Let $\nu^{(j)}$, $j = 1,2,3,4$, be measures on $\mathbb{R}^d$ with $\nu^{(j)}(\{0\}) = 0$ satisfying

$$\begin{aligned}
\nu^{(1)}(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar{f}_{p,\alpha}(s)x)\nu(dx),\\
\nu^{(2)}(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar{f}_{q,\alpha-p}(s)x)\nu^{(1)}(dx),\\
\nu^{(3)}(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar{f}_{q,\alpha-p}(s)x)\nu(dx),\\
\nu^{(4)}(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar{f}_{p,\alpha}(s)x)\nu^{(3)}(dx)
\end{aligned}$$

for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Then

$$\begin{aligned}
\nu^{(2)}(B) &= c_q \int_0^1 (1-t)^{q-1} t^{-\alpha+p-1} dt \int_{\mathbb{R}^d} 1_B(tx)\nu^{(1)}(dx)\\
&= c_q c_p \int_0^1 (1-t)^{q-1} t^{-\alpha+p-1} dt \int_0^1 (1-u)^{p-1} u^{-\alpha-1} du \int_{\mathbb{R}^d} 1_B(tux)\nu(dx)\\
&= c_q c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^1 (1-t)^{q-1} t^{-\alpha+p-1} dt \int_0^1 1_B(tux)(1-u)^{p-1} u^{-\alpha-1} du\\
&= c_q c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^1 (1-t)^{q-1} dt \int_0^t 1_B(wx)(t-w)^{p-1} w^{-\alpha-1} dw\\
&= c_q c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^1 1_B(wx) w^{-\alpha-1} dw \int_w^1 (1-t)^{q-1}(t-w)^{p-1} dt\\
&= c_q c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^1 1_B(wx)(1-w)^{p+q-1} w^{-\alpha-1} dw \int_0^1 (1-y)^{q-1} y^{p-1} dy\\
&\qquad\qquad\qquad\qquad \text{(by change of variables } y = (t-w)/(1-w))\\
&= c_{p+q} \int_0^1 (1-w)^{p+q-1} w^{-\alpha-1} dw \int_{\mathbb{R}^d} 1_B(wx)\nu(dx).
\end{aligned}$$

Hence it follows from Definition 3.25 that

$$\nu^{(2)} \in \mathfrak{M}^L(\mathbb{R}^d) \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\bar{\Phi}^L_{p+q,\alpha}).$$

On the other hand,

$$\nu^{(2)} \in \mathfrak{M}^L(\mathbb{R}^d) \quad \Leftrightarrow \quad \nu^{(1)} \in \mathfrak{D}(\bar{\Phi}^L_{q,\alpha-p})$$

and

$$\nu \in \mathfrak{D}(\bar{\Phi}^L_{p+q,\alpha}) \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$$

by Proposition 4.6. Hence

$$\nu \in \mathfrak{D}(\bar{\Phi}^L_{p+q,\alpha}) \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\bar{\Phi}^L_{p,\alpha}),\ \bar{\Phi}^L_{p,\alpha}\nu \in \mathfrak{D}(\bar{\Phi}^L_{q,\alpha-p})$$

and $\bar{\Phi}^L_{p+q,\alpha} = \bar{\Phi}^L_{q,\alpha-p}\bar{\Phi}^L_{p,\alpha}$. Similarly, in order to see $\bar{\Phi}^L_{p+q,\alpha} = \bar{\Phi}^L_{p,\alpha}\bar{\Phi}^L_{q,\alpha-p}$, observe that

$$\begin{aligned}\nu^{(4)}(B) &= c_p \int_0^1 (1-u)^{p-1} u^{-\alpha-1} du \int_{\mathbb{R}^d} 1_B(ux)\nu^{(3)}(dx)\\ &= c_p c_q \int_0^1 (1-u)^{p-1} u^{-\alpha-1} du \int_0^1 (1-t)^{q-1} t^{-\alpha+p-1} dt \int_{\mathbb{R}^d} 1_B(utx)\nu(dx),\end{aligned}$$

which equals $\nu^{(2)}(B)$ in the preceding calculus. □

Corollary 4.8. *We have*

$$\mathfrak{R}(\bar{\Phi}^L_{p,\alpha}) \supset \mathfrak{R}(\bar{\Phi}^L_{p',\alpha}) \quad \text{for } 0 < p < p' < \infty \text{ and } -\infty < \alpha < 2, \tag{4.31}$$

$$\mathfrak{R}(\bar{\Phi}^L_{\alpha-\beta,\alpha}) \supset \mathfrak{R}(\bar{\Phi}^L_{\alpha'-\beta,\alpha'}) \quad \text{for } -\infty < \beta < \alpha < \alpha' < 2. \tag{4.32}$$

Proof. The decrease (4.31) follows from $\bar{\Phi}^L_{p',\alpha} = \bar{\Phi}^L_{p,\alpha}\bar{\Phi}^L_{p'-p,\alpha-p}$ in (4.30). If $-\infty < \beta < \alpha < \alpha' < 2$, then $\bar{\Phi}^L_{\alpha'-\beta,\alpha'} = \bar{\Phi}^L_{\alpha-\beta,\alpha}\bar{\Phi}^L_{\alpha'-\alpha,\alpha'}$. Hence the decrease (4.32) follows. □

4.3 Range of $\bar{\Phi}^L_{p,\alpha}$

Let us give the description of $\mathfrak{R}(\bar{\Phi}^L_{p,\alpha})$ and the one-to-one property of $\bar{\Phi}^L_{p,\alpha}$.

Theorem 4.9. *Let $-\infty < \alpha < 2$ and $0 < p < \infty$.*

(i) *Let $\nu \in \mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$ with a radial decomposition $(\lambda(d\xi), \nu_\xi)$ and let $\widetilde{\nu} = \bar{\Phi}^L_{p,\alpha}\nu$. Then $\widetilde{\nu}$ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$, where*

$$k_\xi(u) = c_p \int_{(u,\infty)} (r-u)^{p-1} r^{\alpha-p+1} \nu_\xi(dr). \tag{4.33}$$

(ii) *$\bar{\Phi}^L_{p,\alpha}$ is one-to-one.*

Proof. (i) Beginning with (4.29) we have, for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$,

$$\begin{aligned}\widetilde{\nu}(B) &= c_p \int_S \lambda(d\xi) \int_{(0,\infty)} \nu_\xi(dr) \int_0^1 (1-t)^{p-1} t^{-\alpha-1} 1_B(tr\xi) dt\\ &= c_p \int_S \lambda(d\xi) \int_{(0,\infty)} r^{\alpha-p+1}\nu_\xi(dr) \int_0^r (r-u)^{p-1} u^{-\alpha-1} 1_B(u\xi) du\\ &= \int_S \lambda(d\xi) \int_{(0,\infty)} 1_B(u\xi) u^{-\alpha-1} k_\xi(u) du,\end{aligned}$$

where $k_\xi(u)$ is given by (4.33). Since $\nu_\xi(\mathbb{R}_+^\circ) > 0$ for each ξ by the definition of a radial decomposition, $k_\xi(u)$ is not identically zero for each ξ.

(ii) Let $\widetilde{\nu} \in \mathfrak{R}(\bar{\Phi}_{p,\alpha}^L)$. Let $\widetilde{\nu} = \bar{\Phi}_{p,\alpha}^L \nu = \bar{\Phi}_{p,\alpha}^L \nu'$ for some $\nu, \nu' \in \mathfrak{D}(\bar{\Phi}_{p,\alpha}^L)$. Let $(\lambda(d\xi), \nu_\xi)$ and $(\lambda'(d\xi), \nu'_\xi)$ be radial decompositions of ν and ν', respectively. Then $\widetilde{\nu}$ has radial decompositions $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ and $(\lambda'(d\xi), u^{-\alpha-1} k'_\xi(u)du)$, where $k_\xi(u)$ is given by (4.33) and $k'_\xi(u)$ is given by (4.33) with ν'_ξ in place of ν_ξ. Hence, by Proposition 3.1, there is a measurable function $c(\xi)$ satisfying (3.4), (3.5), and $u^{-\alpha-1}k'_\xi(u)du = c(\xi)u^{-\alpha-1}k_\xi(u)du$ for λ-a. e. ξ. Hence

$$k'_\xi(u)du = c(\xi)k_\xi(u)du, \quad \lambda\text{-a. e. } \xi.$$

Since $\widetilde{\nu} \in \mathfrak{M}^L(\mathbb{R}^d)$, $k_\xi(u)du$ and $k'_\xi(u)du$ are, for λ-a. e. ξ, locally finite measures on $\mathbb{R}_+^\circ$. Therefore, Theorem 2.10 on the one-to-one property of I_+^p guarantees that

$$r^{\alpha-p+1}\nu'_\xi(dr) = c(\xi)r^{\alpha-p+1}\nu_\xi(dr), \quad \lambda\text{-a. e. } \xi.$$

Hence $\nu'_\xi = c(\xi)\nu_\xi(dr)$, λ-a. e. ξ, and we obtain $\nu' = \nu$. □

The range of $\bar{\Phi}_{p,\alpha}^L$ is characterized, using the notion of monotonicity of order p.

Theorem 4.10. *Let $-\infty < \alpha < 2$ and $0 < p < \infty$. A measure η on $\mathbb{R}^d$ belongs to $\mathfrak{R}(\bar{\Phi}_{p,\alpha}^L)$ if and only if η is in $\mathfrak{M}^L$ and has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ such that*

$$\begin{aligned}&k_\xi(u) \text{ is measurable in } (\xi,u) \text{ and, for } \lambda\text{-a. e. } \xi,\\ &\text{monotone of order } p \text{ on } \mathbb{R}_+^\circ \text{ in } u.\end{aligned} \tag{4.34}$$

Proof. Let $\eta \in \mathfrak{R}(\bar{\Phi}_{p,\alpha}^L)$. Then $\eta \in \mathfrak{M}^L$ and $\eta = \bar{\Phi}_{p,\alpha}^L \nu$ for some $\nu \in \mathfrak{D}(\bar{\Phi}_{p,\alpha}^L)$. Thus by Theorem 4.9 η has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ with $k_\xi(u)$ of (4.33). Since $\{\nu_\xi\}$ is a measurable family, $k_\xi(u)$ is measurable in (ξ, u). Since $k_\xi(u)du$ and $r^{\alpha-p+1}\nu_\xi(dr)$ are locally finite measures on $\mathbb{R}_+^\circ$ for λ-a. e. ξ, (4.33) shows that $r^{\alpha-p+1}\nu_\xi(dr) \in \mathfrak{D}(I_+^p)$ for λ-a. e. ξ and that $k_\xi(u)$ is monotone of order p on $\mathbb{R}_+^\circ$ for λ-a. e. ξ.

Conversely, suppose that $\eta \in \mathfrak{M}^L$ with a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ satisfying (4.34). Modifying $k_\xi(u)$ for ξ in a λ-null set, we can assume that, for all ξ, $k_\xi(u)$ is monotone of order p on $\mathbb{R}_+^\circ$. From the definition of monotonicity of order p, there is a measure $\sigma_\xi \in \mathfrak{M}_\infty^{p-1}(\mathbb{R}_+^\circ)$ such that

$$k_\xi(u) = c_p \int_{(u,\infty)} (r-u)^{p-1}\sigma_\xi(dr).$$

It follows from Proposition 2.16 that $\{\sigma_\xi\colon \xi \in S\}$ is a measurable family. Let $\nu_\xi(dr) = r^{-\alpha+p-1}\sigma_\xi(dr)$. Define ν by $\nu(B) = \int_S \lambda(d\xi) \int_{(0,\infty)} 1_B(r\xi)\nu_\xi(dr)$. Then the equalities in the proof of Theorem 4.9 (i) show that

$$\eta(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar{f}_{p,\alpha}(s)x)\nu(dx)$$

for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Since $\eta \in \mathfrak{M}^L$, it follows from Remark 3.26 that $\nu \in \mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$ and $\eta = \bar{\Phi}^L_{p,\alpha}\nu$. □

4.4 Classes $K_{p,\alpha}$, $K^0_{p,\alpha}$, and $K^{\mathrm{e}}_{p,\alpha}$

For $-\infty < \alpha < 2$ and $p > 0$ we define

$$K_{p,\alpha} = K_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}(\bar{\Phi}_{p,\alpha}), \tag{4.35}$$

$$K^0_{p,\alpha} = K^0_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}^0(\bar{\Phi}_{p,\alpha}), \tag{4.36}$$

$$K^{\mathrm{e}}_{p,\alpha} = K^{\mathrm{e}}_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}^{\mathrm{e}}(\bar{\Phi}_{p,\alpha}). \tag{4.37}$$

Proposition 4.11. *We have*

$$K^0_{p,\alpha} = K_{p,\alpha} = K^{\mathrm{e}}_{p,\alpha} \quad \text{for } -\infty < \alpha < 1, \tag{4.38}$$

$$K^0_{p,1} \subset K_{p,1} \subset K^{\mathrm{e}}_{p,1}, \tag{4.39}$$

$$K^0_{p,\alpha} = K_{p,\alpha} \subset K^{\mathrm{e}}_{p,\alpha} \quad \text{for } 1 < \alpha < 2. \tag{4.40}$$

Proof. Use Proposition 4.6. If $\alpha < 0$, then (4.38) comes from $\bar{a}_{p,\alpha} < \infty$. If $0 \le \alpha < 1$, than (4.38) comes from Proposition 4.5. We have (4.39) from (3.28). If $1 < \alpha < 2$, then we have (4.40) from (3.28) and Theorem 4.2. □

In Section 5 of [42] it is conjectured that, in the notation of the present article,

$$\bar{\Phi}_{p+q,\alpha} = \bar{\Phi}_{q,\alpha-p}\bar{\Phi}_{p,\alpha} = \bar{\Phi}_{p,\alpha}\bar{\Phi}_{q,\alpha-p} \tag{4.41}$$

for $\alpha \in \mathbb{R}$, $p > 0$, and $q > 0$. For $\rho \in \mathfrak{D}^0(\bar{\Phi}_{p+q,\alpha})$ these equalities are proved.

Theorem 4.12. *Let $-\infty < \alpha < 2$, $p > 0$, and $q > 0$. Let $\rho \in \mathfrak{D}^0(\bar{\Phi}_{p+q,\alpha}) = \mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$. Then*

$$\rho \in \mathfrak{D}^0(\bar{\Phi}_{q,\alpha-p}),\ \bar{\Phi}_{q,\alpha-p}\rho \in \mathfrak{D}^0(\bar{\Phi}_{p,\alpha}),\ \bar{\Phi}_{p,\alpha}\rho \in \mathfrak{D}^0(\bar{\Phi}_{q,\alpha-p}), \tag{4.42}$$

and

$$\bar{\Phi}_{p+q,\alpha}\,\rho = \bar{\Phi}_{q,\alpha-p}\bar{\Phi}_{p,\alpha}\,\rho = \bar{\Phi}_{p,\alpha}\bar{\Phi}_{q,\alpha-p}\,\rho. \tag{4.43}$$

Proof. A distribution $\rho \in ID$ is in $\mathfrak{D}^0(\bar{\Phi}_{p+q,\alpha})$ if and only if

$$c_{p+q}\int_0^1 |C_\rho(tz)|(1-t)^{p+q-1}t^{-\alpha-1}dt < \infty. \tag{4.44}$$

Thus, (4.42) holds if and only if

$$c_q \int_0^1 |C_\rho(tz)|(1-t)^{q-1} t^{-\alpha+p-1} dt < \infty, \tag{4.45}$$

$$c_p \int_0^1 |C_{\bar{\Phi}_{q,\alpha-p}\rho}(tz)|(1-t)^{p-1} t^{-\alpha-1} dt < \infty, \tag{4.46}$$

$$c_q \int_0^1 |C_{\bar{\Phi}_{p,\alpha}\rho}(tz)|(1-t)^{q-1} t^{-\alpha+p-1} dt < \infty. \tag{4.47}$$

We assume $\rho \in \mathfrak{D}^0(\bar{\Phi}_{p+q,\alpha}) = \mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$, that is, (4.44). Then (4.45) holds, since $\int_{1/2}^1 |C_\rho(tz)|(1-t)^{q-1} dt < \infty$ as $|C_\rho(tz)|$ is bounded and since $\int_0^{1/2} |C_\rho(tz)| t^{-\alpha+p-1} dt < \infty$ from (4.44). To see (4.46), notice that the quantity in (4.46) is

$$\begin{aligned}
&= c_p c_q \int_0^1 (1-u)^{p-1} u^{-\alpha-1} du \left| \int_0^1 C_\rho(tuz)(1-t)^{q-1} t^{-\alpha+p-1} dt \right| \\
&\le c_p c_q \int_0^1 (1-u)^{p-1} u^{-\alpha-1} du \int_0^1 |C_\rho(tuz)|(1-t)^{q-1} t^{-\alpha+p-1} dt \\
&= c_{p+q} \int_0^1 |C_\rho(vz)|(1-v)^{p+q-1} v^{-\alpha-1} dv,
\end{aligned}$$

where the last equality is obtained in the proof of Theorem 4.7. (4.47) is similarly true, since the quantity in (4.47) is

$$\le c_q c_p \int_0^1 (1-t)^{q-1} t^{-\alpha+p-1} dt \int_0^1 |C_\rho(utz)|(1-u)^{p-1} u^{-\alpha-1} du.$$

Hence (4.42) is true. Now, the estimate above guarantees the use of Fubini's theorem in showing that

$$\begin{aligned}
c_p c_q \int_0^1 (1-u)^{p-1} u^{-\alpha-1} du \int_0^1 C_\rho(tuz)(1-t)^{q-1} t^{-\alpha+p-1} dt \\
= c_{p+q} \int_0^1 C_\rho(vz)(1-v)^{p+q-1} v^{-\alpha-1} dv.
\end{aligned}$$

Thus we obtain (4.43). □

Remark 4.13. By the method of the proof of Theorem 7.3 (ii) in Section 7.1,we can prove that (4.41) holds if $-\infty < \alpha < 1$, $p > 0$, and $q > 0$, or if $1 < \alpha < 2$, $0 < p < \alpha - 1$, and $q > 0$. □

Now we present some decrease properties for $K_{p,\alpha}$, $K^0_{p,\alpha}$, and $K^{\mathrm{e}}_{p,\alpha}$.

Corollary 4.14. (i) *For* $0 < p < p' < \infty$ *and* $-\infty < \alpha < 2$,

$$K^0_{p,\alpha} \supset K^0_{p',\alpha} \quad \textit{and} \quad K^{\mathrm{e}}_{p,\alpha} \supset K^{\mathrm{e}}_{p',\alpha}. \tag{4.48}$$

(ii) *For* $-\infty < \beta < \alpha < \alpha' < 2$,

$$K_{\alpha-\beta,\alpha} \supset K_{\alpha'-\beta,\alpha'}, \quad K^0_{\alpha-\beta,\alpha} \supset K^0_{\alpha'-\beta,\alpha'}, \quad \textit{and} \quad K^{\mathrm{e}}_{\alpha-\beta,\alpha} \supset K^{\mathrm{e}}_{\alpha'-\beta,\alpha'}. \tag{4.49}$$

Proof. Concerning $K^0_{p,\alpha}$, use Theorem 4.12 and proceed as in the proof of Corollary 4.8. Concerning $K^{\mathrm{e}}_{p,\alpha}$, use Proposition 3.27 and Corollary 4.8. Concerning $K_{\alpha-\beta,\alpha} \supset K_{\alpha'-\beta,\alpha'}$ in (ii), it is a consequence of $K^0_{\alpha-\beta,\alpha} \supset K^0_{\alpha'-\beta,\alpha'}$ if $\alpha \neq 1$ and $\alpha' \neq 1$, since we have Proposition 4.11. If $\alpha = 1$, then $\alpha' > 1$ and $K_{1-\beta,1} \supset K^0_{1-\beta,1} \supset K^0_{\alpha'-\beta,\alpha'} = K_{\alpha'-\beta,\alpha'}$. If $\alpha' = 1$, then $\alpha < 1$ and $K_{\alpha-\beta,\alpha} = K^{\mathrm{e}}_{\alpha-\beta,\alpha} \supset K^{\mathrm{e}}_{1-\beta,1} \supset K_{1-\beta,1}$. □

The decrease property in (i) is true also for $K_{p,\alpha}$, but we have to use the later Theorem 4.18.

Characterization of $K^{\mathrm{e}}_{p,\alpha}$ is as follows.

Theorem 4.15. *Let $-\infty < \alpha < 2$ and $p > 0$. Then $\mu \in K^{\mathrm{e}}_{p,\alpha}$ if and only if $\mu \in ID$ and its Lévy measure ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ satisfying*

$$\begin{aligned} &k_\xi(u) \textit{ is measurable in } (\xi,u) \textit{ and, for } \lambda\textit{-a. e. } \xi, \\ &\textit{monotone of order } p \textit{ on } \mathbb{R}_+^\circ \textit{ in } u. \end{aligned} \tag{4.50}$$

Proof. This follows from Proposition 3.27 and Theorem 4.10 immediately. □

Proposition 4.16. *Let $0 < \alpha < 2$, $p > 0$, and $\mu \in K^{\mathrm{e}}_{p,\alpha}$. Then*

$$\int_{|x|>1} |x|^\beta \mu(dx) < \infty \quad \textit{for all } \beta \in (0,\alpha). \tag{4.51}$$

Proof. The Lévy measure ν_μ is in $\mathfrak{R}(\bar{\Phi}^L_{p,\alpha})$. So $\nu_\mu = \bar{\Phi}^L_{p,\alpha}\nu$ for some $\nu \in \mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$ and

$$\begin{aligned} \int_{|x|>1} |x|^\beta \nu_\mu(dx) &= c_p \int_0^1 (1-t)^{p-1} t^{-\alpha-1} dt \int_{|tx|>1} |tx|^\beta \nu(dx) \\ &= c_p \int_{|x|>1} |x|^\beta \nu(dx) \int_{1/|x|}^1 (1-t)^{p-1} t^{-1+\beta-\alpha} dt \\ &\leq \mathrm{const} \int_{|x|>1} |x|^\alpha \nu(dx) < \infty \end{aligned}$$

from Theorem 4.1. Hence we have (4.51). □

Remark 4.17. Let $0 < \alpha < 2$ and $p > 0$.

(i) There is $\mu \in K^0_{p,\alpha}$ such that $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) = \infty$.

(ii) There is $\mu \in K^0_{p,\alpha}$ which is not Gaussian and satisfies $\int_{\mathbb{R}^d} |x|^{\alpha'} \mu(dx) < \infty$ for all $\alpha' > 0$.

These facts follow from Proposition 5.13 combined with Theorem 5.11 of the later section. □

Characterization of $K_{p,\alpha}$ is as follows.

Theorem 4.18. *Let* $-\infty < \alpha < 2$ *and* $p > 0$. *Let* $\mu \in ID$.

(i) *Assume that* $\alpha < 1$. *Then* $\mu \in K_{p,\alpha}$ *if and only if* ν_μ *has a radial decomposition* $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ *satisfying* (4.50).

(ii) *Assume that* $\alpha = 1$. *Then* $\mu \in K_{p,1}$ *if and only if* μ *has the following two properties:* ν_μ *has a radial decomposition* $(\lambda(d\xi), u^{-2}k_\xi(u)du)$ *satisfying* (4.50) *with* $\alpha = 1$ *and* μ *has weak mean* 0.

(iii) *Assume that* $1 < \alpha < 2$. *Then* $\mu \in K_{p,\alpha}$ *if and only if* μ *has the following two properties:* ν_μ *has a radial decomposition* $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ *satisfying* (4.50) *and* μ *has mean* 0.

Proof. (i) ($\alpha < 1$) Recall (4.38). Then the assertion follows from Proposition 3.27 combined with Theorem 4.10.

(ii) ($\alpha = 1$) Let $f = \bar{f}_{p,1}$. *The "only if" part*. Assume that $\mu \in K_{p,1}$. There is $\rho \in \mathfrak{D}(\bar{\Phi}_{p,1})$ such that $\mu = \bar{\Phi}_{p,1}\rho$. We have $\mu \in K^{\mathrm{e}}_{p,1}$ from (4.39). Hence ν_μ has a radial decomposition $(\lambda(d\xi), u^{-2}k_\xi(u)du)$ with $k_\xi(u)$ satisfying (4.50) with $\alpha = 1$ by Theorem 4.15. We have $\int_{|x|>1}|x|\nu_\rho(dx) < \infty$ and $\int_{\mathbb{R}^d} x\rho(dx) = 0$ from Theorem 4.2 (iii). Hence $\gamma_\rho = -\int_{|x|>1} x\,\nu_\rho(dx)$ from Lemma 4.3. This, combined with (3.22) and (3.35), gives

$$-\gamma_\mu = \lim_{t\to\infty}\int_0^t ds\int_{|f(s)x|>1} f(s)x\,\nu_\rho(dx). \tag{4.52}$$

Hence

$$-\gamma_\mu = \lim_{\varepsilon\downarrow 0} J_\varepsilon,\ \text{where } J_\varepsilon = c_p\int_\varepsilon^1 (1-t)^{p-1}t^{-1}dt\int_{|x|>1/t} x\,\nu_\rho(dx). \tag{4.53}$$

The statement that μ has weak mean 0 is equivalent to the statement that

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon \text{ exists and equals } -\gamma_\mu, \text{ where } I_\varepsilon = \int_{1<|x|\le 1/\varepsilon} x\nu_\mu(dx) \tag{4.54}$$

(see Proposition 3.9). Using a radial decomposition $(\lambda_\rho(d\xi), \nu^\rho_\xi)$ of ν_ρ, we obtain from (3.34) that

$$\begin{aligned} I_\varepsilon &= c_p\int_0^1 (1-t)^{p-1}t^{-2}dt\int_{1<|tx|\le 1/\varepsilon} tx\nu_\rho(dx) \\ &= c_p\int_S \xi\lambda_\rho(d\xi)\int_0^1 (1-t)^{p-1}t^{-1}dt\int_{(1/t,1/(\varepsilon t)]} r\nu^\rho_\xi(dr) \\ &= c_p\int_S \xi\lambda_\rho(d\xi)\int_{(1,\infty)} r\nu^\rho_\xi(dr)\int_{1/r}^{1\wedge 1/(\varepsilon r)} (1-t)^{p-1}t^{-1}dt. \end{aligned}$$

On the other hand,

$$J_\varepsilon = c_p\int_S \xi\lambda_\rho(d\xi)\int_\varepsilon^1 (1-t)^{p-1}t^{-1}dt\int_{(1/t,\infty)} r\nu^\rho_\xi(dr)$$

$$= c_p \int_S \xi \lambda_\rho(d\xi) \int_{(1,\infty)} r\nu_\xi^\rho(dr) \int_{\varepsilon\vee(1/r)}^1 (1-t)^{p-1} t^{-1} dt.$$

Now, in order to prove (4.54), it suffices to show

$$I_\varepsilon - J_\varepsilon \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{4.55}$$

We have

$$\begin{aligned} I_\varepsilon - J_\varepsilon &= c_p \int_S \xi \lambda_\rho(d\xi) \int_{(1,\infty)} r\nu_\xi^\rho(dr) \left(\int_{1/r}^{1\wedge 1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^1 \right) (1-t)^{p-1} t^{-1} dt \\ &= c_p \int_S \xi \lambda_\rho(d\xi) \int_{(1,\infty)} r\nu_\xi^\rho(dr) \left(\int_{1/r}^{1\wedge 1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^1 \right) ((1-t)^{p-1} - 1) t^{-1} dt, \end{aligned}$$

since

$$\begin{aligned} \left(\int_{1/r}^{1\wedge 1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^1 \right) t^{-1} dt &= \log(1\wedge 1/(\varepsilon r)) - \log(1/r) + \log(\varepsilon \vee (1/r)) \\ &= \log(1/\varepsilon) + \log(\varepsilon\wedge(1/r)) - \log(1/r) + \log(\varepsilon\vee(1/r)) = 0. \end{aligned}$$

For any fixed $r > 1$

$$\left(\int_{1/r}^{1\wedge 1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^1 \right) ((1-t)^{p-1} - 1) t^{-1} dt \to 0$$

as $\varepsilon \downarrow 0$. Since $\int_0^1 |(1-t)^{p-1} - 1| t^{-1} dt < \infty$ and $\int_S |\xi| \lambda_\rho(d\xi) \int_1^\infty r\nu_\xi^\rho(dr) = \int_{|x|>1} |x| \nu_\rho(dx) < \infty$, we can use the dominated convergence theorem to conclude (4.55).

The "if" part. We have $\nu_\mu \in \mathfrak{R}(\bar{\Phi}_{p,1}^L)$ and μ has weak mean 0. There is $\nu \in \mathfrak{D}(\bar{\Phi}_{p,1}^L)$ such that $\nu_\mu = \bar{\Phi}_{p,1}^L \nu$. We have $\int_{|x|>1} |x| \nu(dx) < \infty$ from Theorem 4.1. We have $0 < \int_0^\infty f(s)^2 ds < \infty$. Define A by $A = (\int_0^\infty f(s)^2 ds)^{-1} A_\mu$ and $\gamma = -\int_{|x|>1} x\nu(dx)$. Choose $\rho \in ID$ having triplet $(A_\rho, \nu_\rho, \gamma_\rho) = (A, \nu, \gamma)$. We claim that $\rho \in \mathfrak{D}(\bar{\Phi}_{p,1})$ and $\bar{\Phi}_{p,1}\rho = \mu$. Since (3.33) and (3.34) hold, it is enough to show (3.35), that is, γ_{μ_t} of (3.22) converges to γ_μ. Hence it is enough to show (4.53). But we have (4.54), since μ has weak mean 0. The argument in the proof of the "only if" part proves (4.55). We obtain (4.53) from (4.54) and (4.55) combined.

(iii) ($1 < \alpha < 2$) *The "only if" part*. Let $f = \bar{f}_{p,\alpha}$. Similarly to the proof of (i) and (ii), ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} k_\xi(u) du)$ satisfying (4.50). We have $\int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty$ and $\int_{\mathbb{R}^d} x\rho(dx) = 0$ from Theorem 4.2 since $\rho \in \mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$. Thus $\gamma_\rho = -\int_{|x|>1} x\,\nu_\rho(dx)$. Hence we have (4.52), from (3.22) and (3.35). It follows from Proposition 4.16 that $\int_{|x|>1} |x|\,\nu_\mu(dx) < \infty$, that is, $\int_0^\infty ds \int_{|f(s)x|>1} |f(s)x| \nu_\rho(dx) < \infty$. Therefore

$$\gamma_\mu = -\int_0^\infty ds \int_{|f(s)x|>1} f(s) x\,\nu_\rho(dx) = -\int_{|x|>1} x\,\nu_\mu(dx).$$

Hence μ has mean 0.

The "if" part. Using the argument above, it is not hard to modify the proof of the "if" part of (ii). $\square$

Corollary 4.19. *For $0 < p < p' < \infty$ and $-\infty < \alpha < 2$,*

$$K_{p,\alpha} \supset K_{p',\alpha}. \tag{4.56}$$

Proof. This follows from Theorem 4.18 and Corollary 2.6. $\square$

The relation of $K_{p,\alpha}$ and $K^{\mathrm{e}}_{p,\alpha}$ is different between in the case $1 < \alpha < 2$ and in the case $\alpha = 1$.

Corollary 4.20. (i) *If $1 < \alpha < 2$, then any $\mu \in K^{\mathrm{e}}_{p,\alpha}$ can be shifted to an element of $K_{p,\alpha}$.*

(ii) *If $\alpha = 1$, then there is $\mu \in K^{\mathrm{e}}_{p,1}$ such that, for any $x \in \mathbb{R}^d$, the shift $\mu * \delta_x$ of μ does not belong to $K_{p,1}$.*

Proof. The assertion (i) is clear from Theorems 4.15 and 4.18. To see (ii), let λ be a finite measure on S such that $\int_S \xi\lambda(d\xi) \neq 0$ and let

$$\nu(B) = \int_S \lambda(d\xi) \int_2^\infty 1_B(r\xi)\frac{dr}{r^2(\log r)^{1+q}}, \quad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$$

with $0 < q \leq 1$. Then $\int_{|x|>2} |x|\nu(dx) < \infty$ and hence $\nu \in \mathfrak{D}(\bar{\Phi}^L_{p,1})$. Let $\widetilde{\nu} = \bar{\Phi}^L_{p,1}\nu$. Let $\mu \in ID$ be such that $\nu_\mu = \widetilde{\nu}$ and A_μ and γ_μ are arbitrary. Then $\mu \in K^{\mathrm{e}}_{p,1}$. But $\mu \notin K_{p,1}$, since

$$\int_{|x|>1/t} x\nu(dx) = \int_S \xi\lambda(d\xi) \int_{1/t}^\infty \frac{dr}{r(\log r)^{1+q}} = q^{-1}\int_S \xi\lambda(d\xi)\,(\log(1/t))^{-q}$$

for $t < 1/2$ and J_ε in (4.53) is not convergent as $\varepsilon \downarrow 0$. $\square$

In order to characterize $K^0_{p,\alpha}$, we have only to deal with the case $\alpha = 1$, since $K^0_{p,\alpha} = K_{p,\alpha}$ for $\alpha \neq 1$.

Theorem 4.21. *Let $p > 0$. Let $\mu \in ID$. Then $\mu \in K^0_{p,1}$ if and only if μ has the following two properties: ν_μ has a radial decomposition $(\lambda(d\xi), u^{-2}k_\xi(u)du)$ satisfying* (4.50) *with $\alpha = 1$ and μ has weak mean* 0 *absolutely.*

Proof. The "only if" part. Assume $\mu \in K^0_{p,1}$, that is, $\mu = \bar{\Phi}_{p,1}\rho$ for some $\rho \in \mathfrak{D}^0(\bar{\Phi}_{p,1})$. Then $\mu \in K_{p,1}$. We have $\nu_\mu \in \mathfrak{R}(\bar{\Phi}^L_{p,1})$ and μ has weak mean 0 from Theorem 4.18. We also have $\int_{|x|>1} |x|\nu_\rho(dx) < \infty$ and $\int_{\mathbb{R}^d} x\rho(dx) = 0$. Hence $\gamma_\rho = -\int_{|x|>1} x\nu_\rho(dx)$. Hence condition (3.37) is written as

$$\int_0^\infty ds \left| f(s)\int_{|f(s)x|>1} x\nu_\rho(dx)\right| < \infty$$

with $f = \overline{f}_{p,1}$, which is equivalent to

$$c_p \int_0^1 (1-t)^{p-1} t^{-1} dt \left| \int_{|x|>1/t} x \nu_\rho(dx) \right| < \infty. \tag{4.57}$$

Let

$$J = c_p \int_0^1 t^{-1} dt \left| \int_{|x|>1/t} x \nu_\rho(dx) \right|.$$

Condition (4.57) is equivalent to $J < \infty$, since

$$\int_{1/2}^1 (1-t)^{p-1} dt \left| \int_{|x|>1/t} x \nu_\rho(dx) \right| \le \int_{1/2}^1 (1-t)^{p-1} dt \int_{|x|>1} |x| \nu_\rho(dx) < \infty.$$

Let $(\overline{\nu}_\rho(dr), \lambda_r^\rho)$ be a spherical decomposition of ν_ρ such that λ_r^ρ, $r \in \mathbb{R}_+^\circ$, are probability measures on S. For each $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$,

$$\begin{aligned}
\nu_\mu(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x) \nu_\rho(dx) = c_p \int_0^1 (1-t)^{p-1} t^{-2} dt \int_{\mathbb{R}^d} 1_B(tx) \nu_\rho(dx) \\
&= c_p \int_0^1 (1-t)^{p-1} t^{-2} dt \int_{\mathbb{R}_+^\circ} \overline{\nu}_\rho(dr) \int_S 1_B(tr\xi) \lambda_r^\rho(d\xi) \\
&= c_p \int_{\mathbb{R}_+^\circ} \overline{\nu}_\rho(dr) \int_0^1 (1-t)^{p-1} t^{-2} dt \int_S 1_B(tr\xi) \lambda_r^\rho(d\xi) \\
&= c_p \int_{\mathbb{R}_+^\circ} r^{2-p} \overline{\nu}_\rho(dr) \int_0^r (r-u)^{p-1} u^{-2} du \int_S 1_B(u\xi) \lambda_r^\rho(d\xi) \\
&= c_p \int_0^\infty u^{-2} du \int_{(u,\infty)} (r-u)^{p-1} r^{2-p} \left(\int_S 1_B(u\xi) \lambda_r^\rho(d\xi) \right) \overline{\nu}_\rho(dr).
\end{aligned}$$

Assuming that $\nu_\rho \neq 0$, define

$$\lambda_u^\mu(E) = c_p \int_{(u,\infty)} (r-u)^{p-1} r^{2-p} \lambda_r^\rho(E) \overline{\nu}_\rho(dr), \quad E \in \mathscr{B}(S).$$

Then $\{\lambda_u^\mu : u \in \mathbb{R}_+^\circ\}$ is a measurable family of measures on S such that $\lambda_u^\mu(S) < \infty$ for a. e. $u > 0$, since $\int_\varepsilon^\infty u^{-2} \lambda_u^\mu(S) du = \int_{|x|>\varepsilon} \nu_\mu(dx) < \infty$ for $\varepsilon > 0$. We have now

$$\nu_\mu(B) = \int_0^\infty u^{-2} du \int_S 1_B(u\xi) \lambda_u^\mu(d\xi), \qquad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}).$$

and

$$\lambda_u^\mu(S) = c_p \int_{(u,\infty)} (r-u)^{p-1} r^{2-p} \overline{\nu}_\rho(dr).$$

We have $\overline{\nu}_\rho(\mathbb{R}_+^\circ) > 0$ from $\nu_\rho \neq 0$. Let $a = \sup\{r \in \mathbb{R}_+^\circ : \overline{\nu}_\rho((a,\infty)) > 0\}$. If $a = \infty$, then $\lambda_u^\mu(S) > 0$ for all $u \in \mathbb{R}_+^\circ$. If $a < \infty$, then $\lambda_u^\mu(S) > 0$ for $u < a$ and $\lambda_u^\mu(S) = 0$ for $u \ge a$, and hence $\nu_\mu(\{|x| \ge a\}) = \int_{[a,\infty)} u^{-2} \lambda_u^\mu(S) du = 0$. Let

$$\overline{\nu}_\mu(du) = u^{-2} 1_{(0,a)}(u)du.$$

Then, redefining λ_u^μ appropriately for u in a $\overline{\nu}_\mu$-null set, we see that $(\overline{\nu}_\mu(du), \lambda_u^\mu)$ is a spherical decomposition of ν_μ. Let

$$I = \int_1^{a\vee 1} u^{-1}du \left| \int_S \xi \lambda_u^\mu(d\xi) \right|.$$

We have

$$\begin{aligned} I &= c_p \int_1^\infty u^{-1}du \left| \int_{(u,\infty)} (r-u)^{p-1} r^{2-p} \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right| \\ &= c_p \int_0^1 t^{-1}dt \left| \int_{(1/t,\infty)} (r-1/t)^{p-1} r^{2-p} \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right| \\ &= c_p \int_0^1 t^{-1}dt \left| \int_{(1/t,\infty)} (1-1/(rt))^{p-1} r \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right|. \end{aligned}$$

We claim that

$$I < \infty \quad \Leftrightarrow \quad J < \infty. \tag{4.58}$$

In order to see this, it is enough to show that

$$\begin{aligned} \int_0^1 t^{-1}dt \Bigg| &\int_{(1/t,\infty)} (1-1/(rt))^{p-1} r \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \\ &- \int_{(1/t,\infty)} r \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \Bigg| < \infty. \end{aligned} \tag{4.59}$$

We have

$$\begin{aligned} &\int_0^1 t^{-1}dt \int_{(1/t,\infty)} |(1-1/(rt))^{p-1} - 1| r \overline{\nu}_\rho(dr) \\ &\quad = \int_{(1,\infty)} r \overline{\nu}_\rho(dr) \int_{1/r}^1 t^{-1} |(1-1/(rt))^{p-1} - 1| dt \\ &\quad = \int_{(1,\infty)} r \overline{\nu}_\rho(dr) \int_1^r u^{-1} |(1-1/u)^{p-1} - 1| du, \end{aligned}$$

which is finite, since $\int_1^2 (1-1/u)^{p-1}du = \int_{1/2}^1 (1-v)^{p-1} v^{-2}dv < \infty$ and $(1-1/u)^{p-1} - 1 \sim -(p-1)/u$, $u \to \infty$. Therefore (4.59) holds. Hence (4.58) is true. Now recall our assumption that $\mu \in K_{p,1}^0$. Then (3.37) is true from Proposition 3.18. Hence $J < \infty$. Hence, under the assumption that $\nu_\rho \neq 0$, $I < \infty$, which means that μ has weak mean in $\mathbb{R}^d$ absolutely. If $\nu_\rho = 0$, then $\nu_\mu = 0$ and, trivially, μ has weak mean 0 absolutely.

The "if" part. Assume that $\nu_\mu \in \mathfrak{R}(\overline{\Phi}_{p,1}^L)$ and that μ has weak mean 0 absolutely. Then μ has weak mean 0. Hence, it follows from Theorem 4.18 that $\mu \in K_{p,1}$. Then the proof of the "only if" part is valid except the first two lines and the last four lines. Assume that $\nu_\rho \neq 0$. Then $I < \infty$ from the assumption that μ has weak mean 0 absolutely. Hence $J < \infty$ from (4.58), and hence $\mu \in K^0_{p,1}$. If $\nu_\rho = 0$, then $\nu_\mu = 0$ and $\mu \in K^0_{p,1}$. □

Let us strengthen Corollaries 4.14 and 4.19.

Theorem 4.22. (i) *Let* $-\infty < \alpha < 2$ *and* $p > 0$. *Then*

$$K_{p,\alpha} \supsetneqq \bigcup_{p' \in (p,\infty)} K_{p',\alpha}, \quad K^0_{p,\alpha} \supsetneqq \bigcup_{p' \in (p,\infty)} K^0_{p',\alpha}, \quad \textit{and}\ K^{\mathrm{e}}_{p,\alpha} \supsetneqq \bigcup_{p' \in (p,\infty)} K^{\mathrm{e}}_{p',\alpha}. \tag{4.60}$$

(ii) *If* $-\infty < \beta < \alpha < 2$, *then*

$$\begin{aligned} &K_{\alpha-\beta,\alpha} \supsetneqq \bigcup_{\alpha' \in (\alpha,2)} K_{\alpha'-\beta,\alpha'}, \quad K^0_{\alpha-\beta,\alpha} \supsetneqq \bigcup_{\alpha' \in (\alpha,2)} K^0_{\alpha'-\beta,\alpha'}, \\ &\textit{and} \quad K^{\mathrm{e}}_{\alpha-\beta,\alpha} \supsetneqq \bigcup_{\alpha' \in (\alpha,2)} K^{\mathrm{e}}_{\alpha'-\beta,\alpha'}. \end{aligned} \tag{4.61}$$

Proof. It remains only to show the inclusions are strict.

(i) Let $a \in \mathbb{R}^\circ_+$ and $k(u) = (a-u)^{p-1} 1_{(0,a)}(u)$. Then $k(u)$ is monotone of order p on $\mathbb{R}^\circ_+$, but not of order p' on $\mathbb{R}^\circ_+$ for any $p' > p$ (Example 2.17 (a)). We have $\int_0^\infty (u^2 \wedge 1) u^{-\alpha-1} k(u) du < \infty$. Let λ be a nonzero finite measure on S. Then the measure ν of polar product type $(\lambda(d\xi), u^{-\alpha-1}k(u)du)$ is in $\mathfrak{R}(\bar{\Phi}^L_{p,\alpha}) \setminus \mathfrak{R}(\bar{\Phi}^L_{p',\alpha})$ for any $p' > p$. This shows the third relation in (4.60) and the first and second for $\alpha < 1$. Noting that $\int_{|x|>1} |x| \nu(dx) < \infty$, consider $\mu \in ID$ with $\nu_\mu = \nu$, A_μ arbitrary, and $\gamma_\mu = -\int_{|x|>1} x\nu(dx)$. Then μ has mean 0 and we obtain the first and second in (4.60) from Theorems 4.18 and 4.21.

(ii) Let $-\infty < \beta < \alpha < \alpha' < 2$. Let us construct a measure ν in $\mathfrak{R}(\bar{\Phi}^L_{\alpha-\beta,\alpha}) \setminus \mathfrak{R}(\bar{\Phi}^L_{\alpha'-\beta,\alpha'})$ independent of α'. For this, let ν be the measure with radial decomposition $(\lambda(d\xi), u^{-\alpha-1}e^{-u}du)$ where λ is a nonzero finite measure. Then $\nu \in \mathfrak{R}(\bar{\Phi}^L_{\alpha-\beta,\alpha})$, since e^{-u} is completely monotone. Define $l(u)$ by $u^{-\alpha-1}e^{-u} = u^{-\alpha'-1}l(u)$. Then $l(u) = u^{\alpha'-\alpha}e^{-u} \to 0$ as $u \downarrow 0$. Hence $l(u)$ is not monotone of finite order on $\mathbb{R}^\circ_+$, as seen from Proposition 2.13 (iv). Hence $\nu \notin \mathfrak{R}(\bar{\Phi}^L_{\alpha'-\beta,\alpha'})$. The rest of the proof is the same as that of (i) □

We add the one-to-one property of $\bar{\Phi}_{p,\alpha}$.

Theorem 4.23. *Let* $-\infty < \alpha < 2$ *and* $p > 0$. *The mapping* $\bar{\Phi}_{p,\alpha}$ *is one-to-one.*

Proof. Suppose that $\rho, \rho' \in \mathfrak{D}(\bar{\Phi}_{p,\alpha})$ satisfy $\bar{\Phi}_{p,\alpha}\rho = \bar{\Phi}_{p,\alpha}\rho'$. Then, by (3.34) of Proposition 3.18, $\bar{\Phi}^L_{p,\alpha}\nu_\rho = \bar{\Phi}^L_{p,\alpha}\nu_{\rho'}$. Hence $\nu_\rho = \nu_{\rho'}$ follows from Theorem 4.9 (ii). We have also $A_\rho = A_{\rho'}$ from (3.33) of Proposition 3.18, since $0 < \int_0^\infty f(s)^2 ds < \infty$, where we write $f = f_{p,\alpha}$. It follows from (3.22), (3.35), and $\nu_\rho = \nu_{\rho'}$ that

$$\lim_{t\to\infty} \int_0^t f(s)ds \left(\gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|f(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\nu_\rho(dx) \right)$$

$$= \lim_{t\to\infty} \int_0^t f(s)ds \left(\gamma_{\rho'} + \int_{\mathbb{R}^d} x(1_{\{|f(s)x|\le 1\}} - 1_{\{|x|\le 1\}}) \nu_\rho(dx) \right).$$

Hence

$$\lim_{t\to\infty} \int_0^t f(s)(\gamma_\rho - \gamma_{\rho'})ds = 0.$$

Recall that $f(s) > 0$ for $0 < s < \bar{a}_{p,\alpha}$. Now we obtain $\gamma_\rho - \gamma_{\rho'} = 0$ irrespective of whether $\int_0^\infty f(s)ds$ is finite or infinite. Therefore $\rho = \rho'$. □

The continuity property of distributions in $K^{\mathrm{e}}_{p,\alpha}$ is as follows.

Theorem 4.24. (i) *Let μ be a nondegenerate distribution in $K^{\mathrm{e}}_{p,\alpha}$ with $p > 0$ and $\alpha \ge 0$. Then μ is absolutely continuous with respect to d-dimensional Lebesgue measure.*

(ii) *Let $\mu = \bar{\Phi}_{p,\alpha}\rho$ with $p > 0$, $\alpha < 0$, and $\rho \in \mathfrak{D}(\bar{\Phi}_{p,\alpha})$. Then ν_μ is a finite measure if and only if ν_ρ is a finite measure. In particular, for any $p > 0$ and $\alpha < 0$, $K_{p,\alpha}$ contains some compound Poisson distribution, which necessarily has a point mass at the origin.*

Here "μ is nondegenerate" means that the support of μ is not a subset of any translation of any $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$. This theorem generalizes the fact in [38] that nondegenerate selfdecomposable distributions on $\mathbb{R}^d$ are absolutely continuous.

Proof of Theorem 4.24. (i) The Lévy measure ν_μ satisfies $\nu_\mu = \bar{\Phi}^L_{p,\alpha}\nu^0$ for some $\nu^0 \in \mathfrak{D}(\bar{\Phi}^L_{p,\alpha})$. Let $(\lambda(d\xi), \nu^0_\xi(dr))$ be a radial decomposition of ν^0 Then ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}k_\xi(u)du)$ satisfying (4.33) (Theorem 4.9). We have

$$\begin{aligned}\int_0^\infty u^{-\alpha-1}k_\xi(u)du &= c_p \int_0^\infty u^{-\alpha-1}du \int_{(u,\infty)} (r-u)^{p-1} r^{\alpha-p+1} \nu^0_\xi(dr)\\ &= c_p \int_{(0,\infty)} r^{\alpha-p+1}\nu^0_\xi(dr) \int_0^r u^{-\alpha-1}(r-u)^{p-1}du = \infty,\end{aligned}$$

since $\alpha \ge 0$ and $\nu^0_\xi(\mathbb{R}^\circ_+) > 0$. That is, ν_μ is radially absolutely continuous and satisfies the divergence condition in the sense of [39]. Hence μ is absolutely continuous on $\mathbb{R}^d$ by Theorem 27.10 of [39].

(ii) We have $\nu_\mu = \bar{\Phi}^L_{p,\alpha}\nu_\rho$. Then it follows from (4.29) that

$$\nu_\mu(\mathbb{R}^d) = c_p \int_0^1 (1-t)^{p-1} t^{-\alpha-1} dt\, \nu_\rho(\mathbb{R}^d) = (\Gamma_{-\alpha}/\Gamma_{p-\alpha})\nu_\rho(\mathbb{R}^d).$$

Hence the assertion is obvious. □

5 One-parameter subfamilies of $\{K_{p,\alpha}\}$

5.1 $K_{p,\alpha}$*,* $K^0_{p,\alpha}$*, and* $K^{\mathrm{e}}_{p,\alpha}$ *for* $p\in(0,\infty)$ *with fixed* α

As is shown in Theorem 4.22, the one-parameter families $\{K_{p,\alpha}\colon p\in(0,\infty)\}$, $\{K^0_{p,\alpha}\colon p\in(0,\infty)\}$, and $\{K^{\mathrm{e}}_{p,\alpha}\colon p\in(0,\infty)\}$ for fixed $\alpha\in(-\infty,2)$ are strictly decreasing as p increases. The limiting classes as $p\to\infty$ are denoted by

$$K_{\infty,\alpha}=\bigcap_{p>0}K_{p,\alpha},\tag{5.1}$$

$$K^0_{\infty,\alpha}=\bigcap_{p>0}K^0_{p,\alpha},\tag{5.2}$$

$$K^{\mathrm{e}}_{\infty,\alpha}=\bigcap_{p>0}K^{\mathrm{e}}_{p,\alpha}.\tag{5.3}$$

In order to analyze these classes, we use the mappings Ψ_α, $\alpha\in\mathbb{R}$, defined in Section 1.6 from $g_\alpha(t)$ and $f_\alpha(s)$. For $\alpha\geq 0$, $f_\alpha(s)$ is positive for all $s>0$. For $\alpha<0$ we have $f_\alpha(s)=0$ for $s\geq\Gamma_{-\alpha}$.

Asymptotic behaviors of $f_\alpha(s)$ are as follows.

Proposition 5.1. *As* $s\downarrow 0$,

$$f_\alpha(s)\sim-\log s\quad\textit{for }\alpha\in\mathbb{R}.\tag{5.4}$$

As $s\to\infty$,

$$f_0(s)\sim\exp(c-s),\tag{5.5}$$

$$f_\alpha(s)\sim(\alpha s)^{-1/\alpha}\quad\textit{for }\alpha>0,\tag{5.6}$$

$$f_1(s)=s^{-1}-s^{-2}\log s+O(s^{-2}),\tag{5.7}$$

where

$$c=\int_1^\infty u^{-1}e^{-u}du-\int_0^1 u^{-1}(1-e^{-u})du.\tag{5.8}$$

Proof. Since $g_\alpha(t)\sim t^{-\alpha-1}e^{-t}$, $t\to\infty$, we have

$$\lim_{s\downarrow 0}\frac{f_\alpha(s)}{\log(1/s)}=\lim_{t\to\infty}\frac{t}{\log(1/g_\alpha(t))}=\lim_{t\to\infty}\frac{1}{t^{-\alpha-1}e^{-t}/g_\alpha(t)}=1,$$

that is, (5.4) holds. To see (5.5), note that

$$g_0(t)=\int_t^1 u^{-1}du+\int_t^1 u^{-1}(e^{-u}-1)du+\int_1^\infty u^{-1}e^{-u}du=-\log t+c+o(1)$$

as $t\downarrow 0$ and hence $s=-\log f_0(s)+c+o(1)$, $s\to\infty$. To see (5.6), see that $g_\alpha(t)=\alpha^{-1}t^{-\alpha}(1+o(1))$, $t\downarrow 0$, equivalently, $s=\alpha^{-1}f_\alpha(s)^{-\alpha}(1+o(1))$, $s\to\infty$.

Assertion (5.7): We have

$$\begin{aligned} g_1(t) &= \int_t^\infty u^{-2}du + \int_t^1 u^{-2}(e^{-u}-1+u)du - \int_t^1 u^{-1}du + \int_1^\infty u^{-2}(e^{-u}-1)du \\ &= t^{-1} + \log t + O(1), \quad t \downarrow 0 \end{aligned}$$

and hence $s = f_1(s)^{-1} + \log f_1(s) + O(1)$, $s \to \infty$, which is written to

$$f_1(s) = s^{-1} + s^{-1} f_1(s) \log f_1(s) + O(s^{-1} f_1(s)), \quad s \to \infty. \tag{5.9}$$

Since $f_1(s) = s^{-1}(1+o(1))$ from (5.6), we obtain from (5.9)

$$f_1(s) = s^{-1} - s^{-2} \log s + o(s^{-2} \log s), \quad s \to \infty.$$

Putting this again in (5.9), we arrive at (5.7). □

We define $f_\alpha(0) = \infty$ for convenience. Then $f_\alpha(s)$ is locally square-integrable on $\mathbb{R}_+$. We have

$$\Psi_\alpha \rho = \mathscr{L}\left(\int_0^{\infty-} f_\alpha(s) dX_s^{(\rho)} \right),$$

that is, $\Psi_\alpha \rho = \Phi_f \rho$ with $f = f_\alpha$ in (3.24) whenever the improper stochastic integral is definable. If $\alpha < 0$, then $\mathfrak{D}^0(\Psi_\alpha) = \mathfrak{D}(\Psi_\alpha) = \mathfrak{D}^{\mathrm{e}}(\Psi_\alpha) = ID$. By virtue of Proposition 5.1, the domains $\mathfrak{D}^0(\Psi_\alpha)$, $\mathfrak{D}(\Psi_\alpha)$, and $\mathfrak{D}^{\mathrm{e}}(\Psi_\alpha)$ are given by Theorems 4.2 and 4.4. Thus $\mathfrak{D}^0(\Psi_\alpha) = \mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$, $\mathfrak{D}(\Psi_\alpha) = \mathfrak{D}(\bar{\Phi}_{p,\alpha})$, and $\mathfrak{D}^{\mathrm{e}}(\Psi_\alpha) = \mathfrak{D}^{\mathrm{e}}(\bar{\Phi}_{p,\alpha})$ for all α and p. For $\alpha \geq 2$, Ψ_α is trivial.

For $\alpha = -1$ we have

$$g_{-1}(t) = e^{-t}, \quad t \geq 0, \qquad f_{-1}(s) = -\log s, \quad 0 < s \leq 1. \tag{5.10}$$

Hence $\Psi_{-1} = \Upsilon$, where Υ is mentioned in Section 1.7.

Define Ψ_α^L as Φ_f^L in Definition 3.25 with $f = f_\alpha$. This means that

$$\Psi_\alpha^L \nu(B) = \int_0^\infty t^{-\alpha-1} e^{-t} dt \int_{\mathbb{R}^d} 1_B(tx) \nu(dx) \tag{5.11}$$

for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Thus Ψ_α^L is an Υ-transformation of [2]. We have $\mathfrak{D}(\Psi_\alpha^L) = \mathfrak{D}(\bar{\Phi}_{p,\alpha}^L)$, which is described by Theorem 4.1.

The following is an important identity given in Theorem 3.1 of Sato [42] with a long proof. This relates Ψ_α with $\bar{\Phi}_{p,\alpha}$.

Theorem 5.2. *If $-\infty < \alpha < 2$ and $0 < p < \infty$, then*

$$\Psi_\alpha = \Psi_{\alpha-p} \bar{\Phi}_{p,\alpha} = \bar{\Phi}_{p,\alpha} \Psi_{\alpha-p}. \tag{5.12}$$

The prototype of this identity is

$$\Psi_0 = \Upsilon \Phi = \Phi \Upsilon$$

given in Barndorff-Nielsen, Maejima, and Sato [1].

We will use the following two related facts.

Theorem 5.3. *Let* $-\infty < \alpha < 2$ *and* $0 < p < \infty$. *Suppose that* $\rho \in \mathfrak{D}^0(\Psi_\alpha)$. *Then* $\rho \in \mathfrak{D}^0(\Psi_{\alpha-p}) \cap \mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$, $\Psi_{\alpha-p}\rho \in \mathfrak{D}^0(\bar{\Phi}_{p,\alpha})$, $\bar{\Phi}_{p,\alpha}\rho \in \mathfrak{D}^0(\Psi_{\alpha-p})$, *and*

$$\Psi_\alpha \rho = \Psi_{\alpha-p}\bar{\Phi}_{p,\alpha}\rho = \bar{\Phi}_{p,\alpha}\Psi_{\alpha-p}\rho. \tag{5.13}$$

This is given in Lemma 3.2 of [42].

Theorem 5.4. *If* $-\infty < \alpha < 2$ *and* $0 < p < \infty$, *then*

$$\Psi_\alpha^L = \Psi_{\alpha-p}^L \bar{\Phi}_{p,\alpha}^L = \bar{\Phi}_{p,\alpha}^L \Psi_{\alpha-p}^L. \tag{5.14}$$

Proof. Let $\nu \in \mathfrak{M}^L$. Let $\nu^{(j)}$, $j = 1,2$, be measures on $\mathbb{R}^d$ with $\nu^{(j)}(\{0\}) = 0$ satisfying

$$\nu^{(1)}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar{f}_{p,\alpha}(s)x)\nu(dx),$$
$$\nu^{(2)}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f_{\alpha-p}(s)x)\nu^{(1)}(dx)$$

for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Then

$$\begin{aligned}
\nu^{(2)}(B) &= \int_0^\infty t^{-\alpha+p-1}e^{-t}dt \int_{\mathbb{R}^d} 1_B(tx)\nu^{(1)}(dx) \\
&= c_p \int_0^\infty t^{-\alpha+p-1}e^{-t}dt \int_0^1 (1-u)^{p-1}u^{-\alpha-1}du \int_{\mathbb{R}^d} 1_B(tux)\nu(dx) \\
&= c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty t^{-\alpha+p-1}e^{-t}dt \int_0^1 1_B(tux)(1-u)^{p-1}u^{-\alpha-1}du \\
&= c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty e^{-t}dt \int_0^t 1_B(vx)(t-v)^{p-1}v^{-\alpha-1}dv \\
&= c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty 1_B(vx)v^{-\alpha-1}dv \int_v^\infty (t-v)^{p-1}e^{-t}dt \\
&= \int_0^\infty v^{-\alpha-1}e^{-v}dv \int_{\mathbb{R}^d} 1_B(vx)\nu(dx).
\end{aligned}$$

Hence

$$\nu^{(2)} \in \mathfrak{M}^L \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\Psi_\alpha^L).$$

On the other hand,

$$\nu^{(2)} \in \mathfrak{M}^L \quad \Leftrightarrow \quad \nu^{(1)} \in \mathfrak{D}(\Psi_{\alpha-p}^L)$$

and

$$\nu \in \mathfrak{D}(\Psi_\alpha^L) \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\bar{\Phi}_{p,\alpha}^L)$$

by Propositions 4.6 and 5.1. Hence

$$\nu \in \mathfrak{D}(\Psi_\alpha^L) \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\bar{\Phi}_{p,\alpha}^L),\ \bar{\Phi}_{p,\alpha}^L \nu \in \mathfrak{D}(\Psi_{\alpha-p}^L)$$

and $\Psi_\alpha^L = \Psi_{\alpha-p}^L \bar{\Phi}_{p,\alpha}^L$. Proof of $\Psi_\alpha^L = \bar{\Phi}_{p,\alpha}^L \Psi_{\alpha-p}^L$ is similar. □

Theorem 5.5. *Let $-\infty < \alpha < 2$.*

(i) *Let $\nu \in \mathfrak{D}(\Psi_\alpha^L)$ with a radial decomposition $(\lambda(d\xi), \nu_\xi)$ and let $\widetilde{\nu} = \Psi_\alpha^L \nu$. Then $\widetilde{\nu}$ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} k_\xi(u)du)$, where*

$$k_\xi(u) = \int_{\mathbb{R}_+^\circ} r^\alpha e^{-u/r} \nu_\xi(dr). \tag{5.15}$$

(ii) *Ψ_α^L is one-to-one.*

Proof. (i) It follows from (5.11) that

$$\begin{aligned}
\widetilde{\nu}(B) &= \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} \nu_\xi(dr) \int_0^\infty t^{-\alpha-1} e^{-t} 1_B(tr\xi) dt \\
&= \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} r^\alpha \nu_\xi(dr) \int_0^\infty u^{-\alpha-1} e^{-u/r} 1_B(u\xi) du \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} du \int_{\mathbb{R}_+^\circ} r^\alpha e^{-u/r} \nu_\xi(dr).
\end{aligned}$$

(ii) This is proved from the uniqueness in Bernstein's theorem on Laplace transforms. See Proposition 4.1 of [42]. □

Proposition 5.6. *Let $-\infty < \alpha < 2$. The mapping Ψ_α is one-to-one.*

This is proved similarly to Theorem 4.23, using the one-to-one property of Ψ_α^L in Theorem 5.5 (ii).

Theorem 5.7. *Let $-\infty < \alpha < 2$. A measure η on $\mathbb{R}^d$ belongs to $\mathfrak{R}(\Psi_\alpha^L)$ if and only if η is in $\mathfrak{M}^L$ and has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} k_\xi(u)du)$ such that*

$$\begin{aligned}
&k_\xi(u) \text{ is measurable in } \xi \text{ and, for } \lambda\text{-a.e. } \xi, \\
&\text{completely monotone on } \mathbb{R}_+^\circ \text{ in } u.
\end{aligned} \tag{5.16}$$

Using Bernstein's theorem, this theorem is proved from Theorem 5.5 as Theorem 4.10 is from Theorem 4.9. In (4.5) of [42] $k_\xi(u)$ is required not to be identically zero in u and to tend to zero as $u \to \infty$, for λ-a. e. ξ. But it is not identically zero automatically from the definition of radial decomposition in Proposition 3.1; it tends to zero automatically from our definition of complete monotonicity in Section 1.5.

The following Theorem 5.8 and Proposition 5.9 are obtained in parallel to Theorem 4.15 and Proposition 4.16. Theorem 5.8 shows that, for $0 < \alpha < 2$, the class $\mathfrak{R}^{\mathrm{e}}(\Psi_\alpha) \cap \{\mu \in ID \colon A_\mu = 0\}$ is identical with the class of tempered α-stable distributions introduced by Rosiński [34]. He studied properties of the associated Lévy processes on $\mathbb{R}^d$ in detail.

Theorem 5.8. *Let $-\infty < \alpha < 2$. Then $\mu \in \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$ if and only if $\mu \in ID$ and ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} k_\xi(u)du)$ satisfying* (5.16).

Proposition 5.9. *Let* $1 < \alpha < 2$. *If* $\mu \in \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$, *then* (4.51) *holds.*

Theorem 5.10. (i) *Let* $-\infty < \alpha < 1$. *Then* $\mathfrak{R}(\Psi_\alpha) = \mathfrak{R}^0(\Psi_\alpha) = \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$.

(ii) *Let* $\alpha = 1$. *Then* $\mu \in \mathfrak{R}(\Psi_1)$ *if and only if* $\mu \in \mathfrak{R}^{\mathrm{e}}(\Psi_1)$ *and* μ *has weak mean* 0.

(iii) *Let* $\alpha = 1$. *Then* $\mu \in \mathfrak{R}^0(\Psi_1)$ *if and only if* $\mu \in \mathfrak{R}^{\mathrm{e}}(\Psi_1)$ *and* μ *has weak mean* 0 *absolutely.*

(iv) *Let* $1 < \alpha < 2$. *Then* $\mathfrak{R}(\Psi_\alpha) = \mathfrak{R}^0(\Psi_\alpha)$; $\mu \in \mathfrak{R}(\Psi_\alpha)$ *if and only if* $\mu \in \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$ *and* μ *has mean* 0.

Proof. We use Proposition 5.1. Assertions (i) and (iv) are proved similarly to Proposition 4.11 and Theorem 4.18 (iii).

(ii) Method of the proof is the same as in Theorem 4.18 (ii). *The "only if" part.* Let $\mu \in \mathfrak{R}(\Psi_1)$. Then $\mu = \Psi_1\rho$ for some $\rho \in \mathfrak{D}(\Psi_1)$. Define I_ε as in (4.54) and J_ε as

$$J_\varepsilon = \int_\varepsilon^\infty t^{-2}e^{-t}dt \int_{|tx|>1} tx\nu_\rho(dx).$$

Then

$$\begin{aligned} I_\varepsilon &= \int_0^\infty t^{-2}e^{-t}dt \int_{1<|tx|\le 1/\varepsilon} tx\nu_\rho(dx) \\ &= \int_S \xi\lambda_\rho(d\xi) \int_0^\infty t^{-1}e^{-t}dt \int_{(1/t,1/(\varepsilon t)]} r\nu_\xi^\rho(dr) \\ &= \int_S \xi\lambda_\rho(d\xi) \int_{(0,\infty)} r\nu_\xi^\rho(dr) \int_{1/r}^{1/(\varepsilon r)} t^{-1}e^{-t}dt, \\ J_\varepsilon &= \int_S \xi\lambda_\rho(d\xi) \int_\varepsilon^\infty t^{-1}e^{-t}dt \int_{(1/t,\infty)} r\nu_\xi^\rho(dr) \\ &= \int_S \xi\lambda_\rho(d\xi) \int_{(0,\infty)} r\nu_\xi^\rho(dr) \int_{\varepsilon\vee(1/r)}^\infty t^{-1}e^{-t}dt. \end{aligned}$$

Hence

$$\begin{aligned} I_\varepsilon - J_\varepsilon &= \int_S \xi\lambda_\rho(d\xi) \int_{(0,\infty)} r\nu_\xi^\rho(dr) \left(\int_{1/r}^{1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^\infty \right) t^{-1}e^{-t}dt \\ &= \int_S \xi\lambda_\rho(d\xi) \int_{(0,\infty)} r\nu_\xi^\rho(dr) \left(\int_{1/r}^{1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^\infty \right) t^{-1}(e^{-t} - 1_{(0,1)}(t))dt, \end{aligned}$$

because, for any $0 < \varepsilon < 1$ and $r > 0$ we can check

$$\left(\int_{1/r}^{1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^\infty \right) t^{-1}1_{(0,1)}(t)dt = 0.$$

For any fixed $r > 0$,

$$I(\varepsilon, r) = \left(\int_{1/r}^{1/(\varepsilon r)} - \int_{\varepsilon\vee(1/r)}^\infty \right) t^{-1}(e^{-t} - 1_{(0,1)}(t))dt \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Now we can apply the dominated convergence theorem. Recall $\int_{|x|>1}|x|\nu_\rho(dx)<\infty$ and use, for $r\geq 1$,

$$\int_0^\infty t^{-1}|e^{-t}-1_{(0,1)}(t)|dt=\int_0^1 t^{-1}(1-e^{-t})dt+\int_1^\infty t^{-1}e^{-t}dt<\infty$$

and, for $0<r<1$ and $0<\varepsilon<1$,

$$\int_{1/r}^{1/(\varepsilon r)}-\int_{\varepsilon\vee(1/r)}^{\infty}=\int_{1/r}^{1/(\varepsilon r)}-\int_{1/r}^{\infty}=-\int_{1/(\varepsilon r)}^{\infty},$$
$$|I(\varepsilon,r)|\leq\int_{1/(\varepsilon r)}^{\infty}t^{-1}e^{-t}dt\leq\int_{1/(\varepsilon r)}^{\infty}e^{-t}dt=e^{-1/(\varepsilon r)}\leq e^{-1/r}.$$

Therefore $I_\varepsilon-J_\varepsilon\to 0$ as $\varepsilon\downarrow 0$. The rest of the proof is similar to that of the "only if" part of Theorem 4.18 (ii). The "if" part is also similar.

(iii) Method is the same as in the proof of Theorem 4.21. *The "only if" part*. Let $\mu\in\mathfrak{R}^0(\Psi_1)$ with $\mu=\Psi_1\rho$, $\rho\in\mathfrak{D}^0(\Psi_1)$. Let $(\overline{\nu}_\rho,\lambda_r^\rho)$ be a spherical decomposition of ν_ρ such that λ_r^ρ, $r\in\mathbb{R}_+^\circ$, are probability measures on S. Then

$$\begin{aligned}\nu_\mu(B)&=\int_0^\infty t^{-2}e^{-t}dt\int_{\mathbb{R}^d}1_B(tx)\nu_\rho(dx)\\&=\int_0^\infty t^{-2}e^{-t}dt\int_{\mathbb{R}_+^\circ}\overline{\nu}_\rho(dr)\int_S 1_B(tr\xi)\lambda_r^\rho(d\xi)\\&=\int_{\mathbb{R}_+^\circ}r\overline{\nu}_\rho(dr)\int_0^\infty u^{-2}e^{-u/r}du\int_S 1_B(u\xi)\lambda_r^\rho(d\xi)\\&=\int_0^\infty u^{-2}du\int_{(0,\infty)}re^{-u/r}\left(\int_S 1_B(u\xi)\lambda_r^\rho(d\xi)\right)\overline{\nu}_\rho(dr).\end{aligned}$$

Assuming that $\nu_\rho\neq 0$, define

$$\lambda_u^\mu(E)=\int_{(0,\infty)}re^{-u/r}\lambda_r^\rho(E)\overline{\nu}_\rho(dr),\quad E\in\mathscr{B}(S).$$

Then $\{\lambda_u^\mu\colon u\in\mathbb{R}_+^\circ\}$ is a measurable family of measures on S such that $\lambda_u^\mu(S)<\infty$ for a.e. $u>0$. Moreover, $\lambda_u^\mu(S)>0$, $u\in\mathbb{R}_+^\circ$. We have

$$\nu_\mu(B)=\int_0^\infty u^{-2}du\int_S 1_B(u\xi)\lambda_u^\mu(d\xi),\qquad B\in\mathscr{B}(\mathbb{R}^d\setminus\{0\}).$$

and, after redefining λ_u^μ appropriately for u in a Lebesgue-null set, $(u^{-2}du,\lambda_u^\mu(d\xi))$ is a spherical decomposition of ν_μ. Let

$$I=\int_1^\infty u^{-2}du\left|\int_S u\xi\lambda_u^\mu(d\xi)\right|,\qquad J=\int_0^\infty t^{-2}e^{-t}dt\left|\int_{|tx|>1}tx\nu_\rho(dx)\right|.$$

Then

$$\begin{aligned} I &= \int_1^\infty u^{-1} du \left| \int_{\mathbb{R}_+^\circ} r e^{-u/r} \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right| \\ &= \int_0^1 t^{-1} dt \left| \int_{\mathbb{R}_+^\circ} r e^{-1/(tr)} \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right|. \end{aligned}$$

Let

$$J' = \int_0^1 t^{-1} dt \left| \int_{|tx|>1} x \nu_\rho(dx) \right| = \int_0^1 t^{-1} dt \left| \int_{(1/t,\infty)} r \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right|.$$

Then J is finite if and only if J' is finite, since

$$\begin{aligned} &\int_1^\infty t^{-1} e^{-t} dt \left| \int_{|x|>1/t} x \nu_\rho(dx) \right| \le \int_1^\infty t^{-1} e^{-t} dt \int_{|x|>1/t} |x| \nu_\rho(dx) \\ &\quad \le \int_{|x|\le 1} |x| \nu_\rho(dr) \int_{1/|x|}^\infty e^{-t} dt + \int_{|x|>1} |x| \nu_\rho(dr) \int_1^\infty e^{-t} dt < \infty. \end{aligned}$$

On the other hand, let

$$I' = \int_0^1 t^{-1} dt \left| \int_{(1/t,\infty)} r e^{-1/(tr)} \overline{\nu}_\rho(dr) \int_S \xi \lambda_r^\rho(d\xi) \right|.$$

Then I is finite if and only if I' is finite, since

$$\int_0^1 t^{-1} dt \int_{(0,1]} r e^{-1/(tr)} \overline{\nu}_\rho(dr) \le \int_0^1 t^{-1} e^{-1/(2t)} dt \int_{(0,1]} r e^{-1/(2r)} \overline{\nu}_\rho(dr) < \infty$$

and

$$\begin{aligned} &\int_0^1 t^{-1} dt \int_{(1,1/t]} r e^{-1/(tr)} \overline{\nu}_\rho(dr) = \int_1^\infty r \overline{\nu}_\rho(dr) \int_0^{1/r} t^{-1} e^{-1/(tr)} dt \\ &\quad = \int_1^\infty r \overline{\nu}_\rho(dr) \int_0^1 u^{-1} e^{-1/u} du < \infty. \end{aligned}$$

Finally we claim that I' is finite if and only if J' is finite. It is enough to show that

$$\int_0^1 t^{-1} dt \left| \int_{(1/t,\infty)} r e^{-1/(tr)} \overline{\nu}_\rho(dr) - \int_{(1/t,\infty)} r \overline{\nu}_\rho(dr) \right| < \infty.$$

This is proved to be true because

$$\begin{aligned} &\int_0^1 t^{-1} dt \int_{(1/t,\infty)} r \left| e^{-1/(tr)} - 1 \right| \overline{\nu}_\rho(dr) = \int_{(1,\infty)} r \overline{\nu}_\rho(dr) \int_{1/r}^1 (1 - e^{-1/(tr)}) t^{-1} dt \\ &\quad = \int_{(1,\infty)} r \overline{\nu}_\rho(dr) \int_1^r (1 - e^{-1/u}) u^{-1} du \le \mathrm{const} \int_{(1,\infty)} r \overline{\nu}_\rho(dr) < \infty, \end{aligned}$$

since $1-e^{-1/u}=O(1/u)$ as $u\to\infty$. The rest and the proof of the “if” part are a simple modification of the proof of Theorem 4.21. $\square$

Now let us express $K_{\infty,\alpha}$, $K^0_{\infty,\alpha}$, and $K^{\mathrm{e}}_{\infty,\alpha}$ by the ranges of Ψ_α.

Theorem 5.11. *Let* $-\infty<\alpha<2$. *Then*

$$K_{\infty,\alpha}=\mathfrak{R}(\Psi_\alpha), \tag{5.17}$$

$$K^0_{\infty,\alpha}=\mathfrak{R}^0(\Psi_\alpha), \tag{5.18}$$

$$K^{\mathrm{e}}_{\infty,\alpha}=\mathfrak{R}^{\mathrm{e}}(\Psi_\alpha). \tag{5.19}$$

Proof. This follows from Theorems 4.15, 4.18, 4.21, 5.8, and 5.10. $\square$

Let us look at the ranges of Ψ_α as a family with parameter α.

Proposition 5.12. *For* $-\infty<\alpha<2$

$$\begin{gathered}\mathfrak{R}(\Psi_\alpha)\supsetneqq\bigcup_{\alpha'\in(\alpha,2)}\mathfrak{R}(\Psi_{\alpha'}),\qquad \mathfrak{R}^0(\Psi_\alpha)\supsetneqq\bigcup_{\alpha'\in(\alpha,2)}\mathfrak{R}^0(\Psi_{\alpha'}),\\ \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)\supsetneqq\bigcup_{\alpha'\in(\alpha,2)}\mathfrak{R}^{\mathrm{e}}(\Psi_{\alpha'}).\end{gathered} \tag{5.20}$$

For $-\infty<\alpha\le 2$

$$\begin{gathered}\bigcap_{\beta\in(-\infty,\alpha)}\mathfrak{R}(\Psi_\beta)\supsetneqq\mathfrak{R}(\Psi_\alpha),\qquad \bigcap_{\beta\in(-\infty,\alpha)}\mathfrak{R}^0(\Psi_\beta)\supsetneqq\mathfrak{R}^0(\Psi_\alpha),\\ \bigcap_{\beta\in(-\infty,\alpha)}\mathfrak{R}^{\mathrm{e}}(\Psi_\beta)\supsetneqq\mathfrak{R}^{\mathrm{e}}(\Psi_\alpha).\end{gathered} \tag{5.21}$$

See Propositions 4.5, 4.8, 4.15, and 4.17 of [42].

Proposition 5.13. *Let* $0<\alpha<2$.
(i) *If* $\mu\in\mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$, *then* $\int_{\mathbb{R}^d}|x|^\beta\mu(dx)<\infty$ *for all* $\beta\in(0,\alpha)$.
(ii) *There is* $\mu\in\mathfrak{R}^0(\Psi_\alpha)$ *such that* $\int_{\mathbb{R}^d}|x|^\alpha\mu(dx)=\infty$.
(iii) *There is* $\mu\in\mathfrak{R}^0(\Psi_\alpha)$ *which is not Gaussian and satisfies, for all* $\alpha'>0$, $\int_{\mathbb{R}^d}|x|^{\alpha'}\mu(dx)<\infty$.
(iv) *There is* $\mu\in\mathfrak{R}^0(\Psi_0)=T$ *such that, for all* $\alpha'>0$, $\int_{\mathbb{R}^d}|x|^{\alpha'}\mu(dx)=\infty$.

For (i)–(iii), see the proof of Proposition 4.10 of [42]. For (iv), see Proposition 4.12 of [42].

The ranges of Ψ_α have the following relations with the classes $\mathfrak{S}_\alpha$ and $\mathfrak{S}^0_\alpha$ of α-stable and strictly α-stable distributions.

Proposition 5.14. (i) *Let* $0<\alpha\le 1$. *We have*

$$\mathfrak{S}_\alpha\subset\bigcap_{\beta\in(0,\alpha)}\mathfrak{R}^0(\Psi_\beta). \tag{5.22}$$

If $\mu \in \mathfrak{S}_\alpha$ and μ is not a δ-measure, then $\mu \notin \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$.

(ii) *Let $1 < \alpha \le 2$. We have*

$$\mathfrak{S}_\alpha^0 \subset \bigcap_{\beta \in (0,\alpha)} \mathfrak{R}^0(\Psi_\beta). \tag{5.23}$$

If $\mu \in \mathfrak{S}_\alpha \setminus \mathfrak{S}_\alpha^0$, then $\mu \notin \bigcup_{\beta \in (1,2]} \mathfrak{R}^0(\Psi_\beta)$. If $\mu \in \mathfrak{S}_\alpha$ and μ is not a δ-measure, then $\mu \notin \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$.

See the proof of Proposition 4.7 of [42].

Remark 5.15. Open question: Do the limiting classes $\bigcap_{\alpha<2} \mathfrak{R}^0(\Psi_\alpha)$ and $\bigcap_{\alpha<2} \mathfrak{R}^{\mathrm{e}}(\Psi_\alpha)$ contain a distribution other than Gaussian? □

The final remark concerns another one-parameter subfamily.

Remark 5.16. As Theorem 4.22 says, the one-parameter families $\{K_{\alpha-\beta,\alpha} \colon \alpha \in (\beta,\infty)\}$, $\{K^0_{\alpha-\beta,\alpha} \colon \alpha \in (\beta,\infty)\}$, and $\{K^{\mathrm{e}}_{\alpha-\beta,\alpha} \colon \alpha \in (\beta,\infty)\}$ are strictly decreasing as α increases. Open question: What are the limiting classes as $\alpha \to \infty$? □

The mapping Ψ_α is extended to the class of mappings $\Psi_{\alpha,\beta}$ with two parameters $-\infty < \alpha < 2$ and $\beta > 0$ by Maejima and Nakahara [22]. Let $G_{\alpha,\beta}(t) = \int_t^\infty u^{-\alpha-1} e^{-u^\beta} du$, $0 < t < \infty$. Define $t = F_{\alpha,\beta}(s)$, $0 < s < G_{\alpha,\beta}(0+)$, by $s = G_{\alpha,\beta}(t)$, $0 < t < \infty$. If $\alpha < 0$, then define $F_{\alpha,\beta}(s) = 0$ for $s \ge G_{\alpha,\beta}(0+)$. Let $\Psi_{\alpha,\beta} = \Phi_f$ in (3.24) with $f = F_{\alpha,\beta}$. They gave representation of Lévy measures for $\mathfrak{R}(\Psi_{\alpha,\beta})$.

5.2 $K_{p,\alpha}$, $K^0_{p,\alpha}$, and $K^{\mathrm{e}}_{p,\alpha}$ for $\alpha \in (-\infty, 2)$ with fixed p

We use the following lemma.

Lemma 5.17. *Let n be a positive integer. If $f_1(r)$ and $f_2(r)$ are monotone of order n on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$], then $f_1(r) f_2(r)$ is monotone of order n on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$].*

Proof. In case $n = 1$, the assertion is obvious from Proposition 2.11 (i). Let $n \ge 2$. Assume that the assertion is true for $n-1$ in place of n. A function $f(r)$ is monotone of order n on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$] if and only if $f(r) = \int_r^\infty \varphi(s) ds$ with a function φ monotone of order $n-1$ on $\mathbb{R}$ [resp. $\mathbb{R}_+^\circ$]. Let f_j, $j = 1, 2$, be monotone of order n. Then $f_j(r) = \int_r^\infty \varphi_j(s) ds$ and

$$\begin{aligned}
\int_r^\infty \varphi_1(s) ds \int_r^\infty \varphi_2(t) dt &= \int_r^\infty \varphi_1(s) ds \int_r^s \varphi_2(t) dt + \int_r^\infty \varphi_1(s) ds \int_s^\infty \varphi_2(t) dt \\
&= \int_r^\infty dt \left(\varphi_2(t) \int_t^\infty \varphi_1(s) ds + \varphi_1(t) \int_t^\infty \varphi_2(s) ds \right),
\end{aligned}$$

which shows that $f_1(r) f_2(r)$ is monotone of order n. □

Lemma 5.18. *Let n be a positive integer. If $f(r)$ is monotone of order n on $\mathbb{R}_+^\circ$, then, for any $a>0$, $r^{-a}f(r)$ is monotone of order n on $\mathbb{R}_+^\circ$.*

Proof. Apply Lemma 5.17 for $f_1 = f$ and $f_2 = r^{-a}$, which is completely monotone on $\mathbb{R}_+^\circ$. ☐

Theorem 5.19. *Let n be a positive integer. Let $-\infty < \alpha < \alpha' < 2$. Then*

$$K_{n,\alpha} \supsetneqq K_{n,\alpha'}, \quad K^0_{n,\alpha} \supsetneqq K^0_{n,\alpha'}, \quad \textit{and} \quad K^{\mathrm{e}}_{n,\alpha} \supsetneqq K^{\mathrm{e}}_{n,\alpha'}. \tag{5.24}$$

Proof. Let us prove $K^{\mathrm{e}}_{n,\alpha} \supsetneqq K^{\mathrm{e}}_{n,\alpha'}$. Let $\mu \in K_{n,\alpha'}$. By Theorem 4.15, ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha'-1}k_\xi(u)du)$ such that $k_\xi(u)$ is measurable in (ξ,u) and $k_\xi(u)$ is monotone of order n on $\mathbb{R}_+^\circ$ in u. Notice that $u^{-\alpha'-1}k_\xi(u) = u^{-\alpha-1}k^\flat_\xi(u)$ with $k^\flat_\xi(u) = u^{\alpha-\alpha'}k_\xi(u)$. It follows from Lemma 5.18 that $k^\flat_\xi(u)$ is monotone of order n on $\mathbb{R}_+^\circ$. Hence $\mu \in K_{n,\alpha}$. Thus $K^{\mathrm{e}}_{n,\alpha} \supset K^{\mathrm{e}}_{n,\alpha'}$. To show the strictness of the inclusion, let λ be a non-zero finite measure on S and let $k(u) = (1-u)^{n-1}1_{(0,1)}(u)$, which is monotone of order n on $\mathbb{R}_+^\circ$ (Example 2.17 (a)). Let ν be the Lévy measure of polar product type $(\lambda(d\xi), u^{-\alpha-1}k(u)du)$. Let $\mu \in ID$ with $\nu_\mu = \nu$. Then $\mu \in K^{\mathrm{e}}_{n,\alpha}$. But $\mu \notin K^{\mathrm{e}}_{n,\alpha'}$, because the function $k^\sharp$ satisfying $u^{-\alpha-1}k(u) = u^{-\alpha'-1}k^\sharp(u)$ is expressed as

$$k^\sharp(u) = u^{\alpha'-\alpha}(1-u)^{n-1}1_{(0,1)}(u),$$

which is not monotone of any order on $\mathbb{R}_+^\circ$ by virtue of Proposition 2.13 (iv). Hence $K^{\mathrm{e}}_{n,\alpha} \setminus K^{\mathrm{e}}_{n,\alpha'} \neq \emptyset$. The first and second relations in (5.24) are obtained from the third by the use of Theorems 4.18 and 4.21. ☐

Remark 5.20. Open question: Is Lemma 5.18 true for $p \in \mathbb{R}_+^\circ$ in place of n? If the answer is affirmative, then Theorem 5.19 is true for $p \in \mathbb{R}_+^\circ$ in place of n. ☐

Iksanov, Jurek, and Schreiber [10] contains the identity

$$\Phi\rho = (\bar{\Phi}_{1,-1}\Phi\rho) * (\bar{\Phi}_{1,-1}\rho) = \bar{\Phi}_{1,-1}((\Phi\rho)*\rho) \quad \text{for } \rho \in \mathfrak{D}(\Phi).$$

This is generalized to the following identity for the family $\{\bar{\Phi}_{1,\alpha} \colon \alpha \in (-\infty,2)\}$. Essentially the same result is given by Czyżewska-Jankowska and Jurek [7]. We use Propositions 3.19 and 3.20.

Theorem 5.21. *Let $-\infty < \alpha < \alpha' < 2$. If $\rho \in \mathfrak{D}^0(\bar{\Phi}_{1,\alpha'})$, then $\rho \in \mathfrak{D}^0(\bar{\Phi}_{1,\alpha})$, $\bar{\Phi}_{1,\alpha'}\rho \in \mathfrak{D}^0(\bar{\Phi}_{1,\alpha})$, and*

$$\bar{\Phi}_{1,\alpha'}\rho = \left(\bar{\Phi}_{1,\alpha}\bar{\Phi}_{1,\alpha'}(\rho^{\alpha'-\alpha})\right) * \bar{\Phi}_{1,\alpha}\rho. \tag{5.25}$$

Proof. Recall that

$$\int_0^\infty |C_\rho(\bar{f}_{1,\alpha}(s)z)|ds = \int_0^1 |C_\rho(tz)|t^{-\alpha-1}dt.$$

Suppose that $\rho \in \mathfrak{D}^0(\bar{\Phi}_{1,\alpha'})$ and let $\mu = \bar{\Phi}_{1,\alpha'}\rho$. Then $\int_0^1 |C_\rho(tz)| t^{-\alpha'-1} dt < \infty$. Hence $\int_0^1 |C_\rho(tz)| t^{-\alpha-1} dt < \infty$, that is, $\rho \in \mathfrak{D}^0(\bar{\Phi}_{1,\alpha})$. Further,

$$\begin{aligned}
\int_0^1 |C_\mu(tz)| t^{-\alpha-1} dt &= \int_0^1 t^{-\alpha-1} dt \left| \int_0^1 C_\rho(stz) s^{-\alpha'-1} ds \right| \\
&= \int_0^1 t^{\alpha'-\alpha-1} dt \left| \int_0^t C_\rho(uz) u^{-\alpha'-1} du \right| \le \int_0^1 t^{\alpha'-\alpha-1} dt \int_0^t |C_\rho(uz)| u^{-\alpha'-1} du \\
&= \int_0^1 |C_\rho(uz)| u^{-\alpha'-1} du \int_u^1 t^{\alpha'-\alpha-1} dt \le (\alpha'-\alpha)^{-1} \int_0^1 |C_\rho(uz)| u^{-\alpha'-1} du < \infty.
\end{aligned}$$

Hence $\mu \in \mathfrak{D}^0(\bar{\Phi}_{1,\alpha})$. Let $\widetilde{\mu} = \bar{\Phi}_{1,\alpha}\mu$. Then

$$\begin{aligned}
C_{\widetilde{\mu}}(z) &= \int_0^1 C_\mu(tz) t^{-\alpha-1} dt = \int_0^1 t^{-\alpha-1} dt \int_0^1 C_\rho(stz) s^{-\alpha'-1} ds \\
&= \int_0^1 t^{\alpha'-\alpha-1} dt \int_0^t C_\rho(uz) u^{-\alpha'-1} du = \int_0^1 C_\rho(uz) u^{-\alpha'-1} du \int_u^1 t^{\alpha'-\alpha-1} dt \\
&= (\alpha'-\alpha)^{-1} \int_0^1 C_\rho(uz)(u^{-\alpha'-1} - u^{-\alpha-1}) du.
\end{aligned}$$

Hence

$$(\alpha'-\alpha) C_{\widetilde{\mu}}(z) + \int_0^1 C_\rho(uz) u^{-\alpha-1} du = \int_0^1 C_\rho(uz) u^{-\alpha'-1} du,$$

which is (5.25). □

The fact $K^0_{1,\alpha} \supset K^0_{1,\alpha'}$ for $-\infty < \alpha < \alpha' < 2$ follows also from the theorem above.

Maejima, Matsui, and Suzuki [21] and Maejima and Ueda [28] studied essentially the same class as $K_{1,\alpha}$ with parameter α. They gave the description of the triplet of $\mu \in K_{1,\alpha}$ and a kind of decomposability which generalizes (1.2), and introduced a generalization of Ornstein–Uhlenbeck type process which corresponds to this class. An earlier paper [14] of Jurek is also related.

6 Second two-parameter extension $L_{p,\alpha}$ of the class L of selfdecomposable distributions

6.1 $\Lambda_{p,\alpha}$ and $\Lambda^L_{p,\alpha}$

For $-\infty < \alpha < \infty$ and $p > 0$ we have introduced $j_{p,\alpha}(t)$, $l_{p,\alpha}(s)$, and $\Lambda_{p,\alpha}$ in Section 1.6. Namely, $j_{p,\alpha}(t)$, $0 < t \le 1$ is defined by (1.14); $b_{p,\alpha} = j_{p,\alpha}(0+)$ equals $(-\alpha)^{-p}$ for $\alpha < 0$ and ∞ for $\alpha \ge 0$; $t = l_{p,\alpha}(s)$, $0 \le s < b_{p,\alpha}$, if and only if $s = j_{p,\alpha}(t)$, $0 < t \le 1$; $l_{p,\alpha}(s)$ is defined to be zero if $\alpha < 0$ and $s \ge b_{p,\alpha}$; $\Lambda_{p,\alpha} = \Phi_f$ with

$f = l_{p,\alpha}$ in (3.24). Define the transformation $\Lambda^L_{p,\alpha}$ of Lévy measures as $\Lambda^L_{p,\alpha} = \Phi^L_f$ in Definition 3.25 with $f = l_{p,\alpha}$.

We note the following special cases. If $p = 1$, then

$$j_{1,\alpha}(t) = \bar{g}_{1,\alpha}(t), \qquad l_{1,\alpha}(s) = \bar{f}_{1,\alpha}(s), \tag{6.1}$$

so that the explicit forms are given in (4.19)–(4.21). Thus $\Lambda_{1,\alpha} = \bar{\Phi}_{1,\alpha}$ and $\Lambda_{1,0} = \Phi$. If $p > 0$ and $\alpha = 0$, then

$$j_{p,0}(t) = c_{p+1}(-\log t)^p, \quad 0 \le t \le 1, \tag{6.2}$$

$$l_{p,0}(s) = \exp(-(\Gamma_{p+1}s)^{1/p}), \quad s \ge 0. \tag{6.3}$$

From the definition of absolute definability we have

$$\rho \in \mathfrak{D}^0(\Lambda_{p,\alpha}) \quad \Leftrightarrow \quad \int_0^1 |C_\rho(tz)|(-\log t)^{p-1} t^{-\alpha-1} dt < \infty.$$

It follows that

$$\mathfrak{D}^0(\Lambda_{p,\alpha}) \supset \mathfrak{D}^0(\Lambda_{p',\alpha}) \quad \text{if } 0 < p < p'. \tag{6.4}$$

Proposition 6.1. *If $\alpha > 0$, then, as $s \to \infty$,*

$$l_{p,\alpha}(s) \sim (\alpha\Gamma_p s)^{-1/\alpha}(\alpha^{-1}\log s)^{(p-1)/\alpha} \quad \textit{for } p > 0. \tag{6.5}$$

Proof. Let $\alpha > 0$. We have

$$j_{p,\alpha}(t) = \alpha^{-1} c_p (-\log t)^{p-1} t^{-\alpha}(1 + o(1)), \qquad t \downarrow 0.$$

Let $s = j_{p,\alpha}(t)$ and $t = l_{p,\alpha}(s) = l(s) = s^{-1/\alpha} l^\sharp(s)$. Then

$$s = \alpha^{-1} c_p (-\log l(s))^{p-1} l(s)^{-\alpha}(1 + o(1)), \qquad s \to \infty. \tag{6.6}$$

If $p = 1$, this shows (6.5). Assume $p \ne 1$ in the following. It follows from (6.6) that

$$l^\sharp(s)^{\alpha/(p-1)} = (\alpha^{-1}c_p)^{1/(p-1)}(\alpha^{-1}\log s - \log l^\sharp(s))(1 + o(1)).$$

Define $l^{\sharp\sharp}(s)$ as $l^\sharp(s) = (\alpha^{-1}c_p)^{1/\alpha}(\alpha^{-1}\log s)^{(p-1)/\alpha} l^{\sharp\sharp}(s)$. Then we see that

$$l^{\sharp\sharp}(s)^{\alpha/(p-1)} = \left(1 - \frac{B}{\log s} - \frac{(p-1)\log\log s}{\log s} - \frac{\alpha \log l^{\sharp\sharp}(s)}{\log s}\right)(1 + o(1)), \tag{6.7}$$

where B is a constant independent of s. Let $s_n \to \infty$ be a sequence such that $l^{\sharp\sharp}(s_n)$ tends to some $C \in [0,\infty]$. If C is 0 or ∞, then we have a contradiction from (6.7) when $p > 1$ as well as when $p < 1$. Hence $C \ne 0, \infty$. Then we obtain $C = 1$ again from (6.7). It follows that $l^{\sharp\sharp}(s) \to 1$ as $s \to \infty$, which shows (6.5). □

Theorem 6.2. *Let $-\infty < \alpha < \infty$ and $p > 0$. The domain of $\Lambda^L_{p,\alpha}$ is as follows:*

$$\mathfrak{D}(\Lambda^L_{p,\alpha}) = \mathfrak{M}^L \quad \textit{if } \alpha < 0, \tag{6.8}$$

$$\mathfrak{D}(\Lambda^L_{p,0}) = \{\nu \in \mathfrak{M}^L \colon \textstyle\int_{|x|>2} (\log|x|)^p \nu(dx) < \infty\} \quad \textit{if } \alpha = 0, \tag{6.9}$$

$$\mathfrak{D}(\Lambda^L_{p,\alpha}) = \{\nu \in \mathfrak{M}^L \colon \textstyle\int_{|x|>2} (\log|x|)^{p-1}|x|^\alpha \nu(dx) < \infty\} \quad \textit{if } 0 < \alpha < 2, \tag{6.10}$$

$$\mathfrak{D}(\Lambda^L_{p,\alpha}) = \{\delta_0\} \quad \textit{if } \alpha \geq 2. \tag{6.11}$$

Recall that $\Lambda^L_{1,\alpha} = \bar{\Phi}^L_{1,\alpha}$ and notice that this theorem for $p = 1$ is consistent with Theorem 4.1.

Proof of Theorem 6.2. Let $0 < \alpha < 2$. Given ν, we express the measure $\widetilde{\nu}$ in Definition 3.25 for $f = l_{p,\alpha}$ as

$$\widetilde{\nu}(B) = c_p \int_0^1 (-\log t)^{p-1} t^{-\alpha-1} dt \int_{\mathbb{R}^d} 1_B(tx)\nu(dx) \tag{6.12}$$

for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. We use the fact that

$$\int_0^u (-\log t)^{p-1} t^q dt \sim (q+1)^{-1}(-\log u)^{p-1} u^{q+1}, \quad u \downarrow 0, \quad \text{for } p \in \mathbb{R},\ q > -1, \tag{6.13}$$

$$\int_u^1 (-\log t)^{p-1} t^q dt \sim (-q-1)^{-1}(-\log u)^{p-1} u^{q+1}, \quad u \downarrow 0, \quad \text{for } p > 0,\ q < -1. \tag{6.14}$$

Thus

$$\begin{aligned}
\int_{|x|\leq 1} |x|^2 \widetilde{\nu}(dx) &= c_p \int_0^1 (-\log t)^{p-1} t^{-\alpha-1} dt \int_{|tx|\leq 1} |tx|^2 \nu(dx) \\
&= c_p \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_0^{1\wedge(1/|x|)} (-\log t)^{p-1} t^{1-\alpha} dt \\
&\leq C_1 \int_{|x|\leq 2} |x|^2 \nu(dx) + C_2 \int_{|x|>2} (\log|x|)^{p-1} |x|^\alpha \nu(dx), \\
\int_{|x|>1} \widetilde{\nu}(dx) &= c_p \int_0^1 (-\log t)^{p-1} t^{-\alpha-1} dt \int_{|tx|>1} \nu(dx) \\
&= c_p \int_{|x|>1} \nu(dx) \int_{1/|x|}^1 (-\log t)^{p-1} t^{-\alpha-1} dt \\
&\leq C_3 \int_{1<|x|\leq 2} \nu(dx) + C_4 \int_{|x|>2} (\log|x|)^{p-1} |x|^\alpha \nu(dx).
\end{aligned}$$

Here $C_1, \ldots, C_4$ are positive constants. Similarly we can show the reverse estimates. Hence (6.10) is true.

If $\alpha < 0$, then (6.8) comes from $b_{p,\alpha} < \infty$. If $\alpha = 0$, then we have (6.9) as in Theorem 5.15 of [41] and Proposition 4.3 of [43]. If $\alpha \geq 2$, then we have (6.11) by a similar argument. □

Let us study the domains of $\Lambda_{p,\alpha}$.

Theorem 6.3. *Let $-\infty < \alpha < \infty$ and $p > 0$.*

(i) *If $\alpha < 0$, then*

$$\mathfrak{D}^0(\Lambda_{p,\alpha}) = \mathfrak{D}(\Lambda_{p,\alpha}) = \mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha}) = ID.$$

(ii) *If $0 \leq \alpha < 2$, then*

$$\mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha}) = \{\rho \in ID\colon\, \nu_\rho \in \mathfrak{D}(\Lambda^L_{p,\alpha})\}.$$

(iii) *If $0 \leq \alpha < 1$, then*

$$\mathfrak{D}^0(\Lambda_{p,\alpha}) = \mathfrak{D}(\Lambda_{p,\alpha}) = \mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha}).$$

(iv) *If $\alpha = 1$ and $p \geq 1$, then*

$$\mathfrak{D}^0(\Lambda_{p,1}) \subsetneqq \mathfrak{D}(\Lambda_{p,1}) \subsetneqq \mathfrak{D}^{\mathrm{e}}(\Lambda_{p,1}),$$

$$\begin{aligned} \mathfrak{D}(\Lambda_{p,1}) = \{\rho \in ID\colon\, &\nu_\rho \in \mathfrak{D}(\Lambda^L_{p,1}),\ \textstyle\int_{\mathbb{R}^d} x\rho(dx) = 0, \\ &\lim_{a\to\infty}\int_{|x|>1} x(\log(|x|\wedge a))^p \nu_\rho(dx) \textit{ exists in } \mathbb{R}^d\}, \end{aligned} \tag{6.15}$$

$$\begin{aligned} \mathfrak{D}^0(\Lambda_{p,1}) = \{\rho \in ID\colon\, &\nu_\rho \in \mathfrak{D}(\Lambda^L_{p,1}),\ \int_{\mathbb{R}^d} x\rho(dx) = 0, \\ &\int_0^1 (-\log t)^{p-1} t^{-1} dt \left| \int_{|x|>1/t} x\nu_\rho(dx)\right| < \infty\}. \end{aligned} \tag{6.16}$$

(v) *If $1 < \alpha < 2$, then*

$$\begin{aligned} \mathfrak{D}^0(\Lambda_{p,\alpha}) = \mathfrak{D}(\Lambda_{p,\alpha}) &= \{\rho \in ID\colon\, \nu_\rho \in \mathfrak{D}(\Lambda^L_{p,\alpha}),\ \textstyle\int_{\mathbb{R}^d} x\rho(dx) = 0\} \\ &\subsetneqq \mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha}). \end{aligned}$$

(vi) *If $\alpha \geq 2$, then*

$$\mathfrak{D}^0(\Lambda_{p,\alpha}) = \mathfrak{D}(\Lambda_{p,\alpha}) = \{\delta_0\} \subsetneqq \mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha}) = \{\delta_\gamma\colon\, \gamma \in \mathbb{R}^d\}.$$

Proof. If $1 < \alpha < 2$ or if $\alpha = 1$ with $p \geq 1$, then $\int_{\mathbb{R}^d} |x|\rho(dx) < \infty$ for ρ satisfying $\nu_\rho \in \mathfrak{D}(\Lambda^L_{p,\alpha})$ (see Theorem 6.2). We write $l(s) = l_{p,\alpha}(s)$ for simplicity. We use $C_1, C_2, \ldots$ for positive constants.

(i) Note that $b_{p,\alpha} < \infty$ for $\alpha < 0$.

(ii) Note that $\int_0^\infty l(s)^2 ds < \infty$ for $0 \leq \alpha < 2$ by (6.5) of Proposition 6.1.

(iii) Let $0 < \alpha < 1$. Let $\rho \in \mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha})$. We have $\int_0^\infty l(s)ds < \infty$ from (6.5). Choosing $s_0 = j_{p,\alpha}(t_0) > 0$ such that $l(s) < 1$ for $s > s_0$, we have

$$
\begin{aligned}
&\int_{s_0}^{\infty} l(s)ds \int_{\mathbb{R}^d} |x(1_{\{|l(s)x|\le 1\}} - 1_{\{|x|\le 1\}})| \nu_\rho(dx) \\
&\quad = \int_{s_0}^{\infty} l(s)ds \left(\int_{|x|\le 1} |x| 1_{\{|l(s)x|>1\}} \nu_\rho(dx) + \int_{|x|>1} |x| 1_{\{|l(s)x|\le 1\}} \nu_\rho(dx) \right) \\
&\quad = \int_{s_0}^{\infty} l(s)ds \int_{|x|>1} |x| 1_{\{|l(s)x|\le 1\}} \nu_\rho(dx) \\
&\quad = c_p \int_0^{t_0} (-\log t)^{p-1} t^{-\alpha} dt \int_{|x|>1} |x| 1_{\{|tx|\le 1\}} \nu_\rho(dx) \\
&\quad = c_p \int_{|x|>1} |x| \nu_\rho(dx) \int_0^{t_0\wedge(1/|x|)} (-\log t)^{p-1} t^{-\alpha} dt \\
&\quad = C_1 \int_{|x|>1} |x| \nu_\rho(dx) + c_p \int_{|x|>1/t_0} |x| \nu_\rho(dx) \int_0^{1/|x|} (-\log t)^{p-1} t^{-\alpha} dt \\
&\quad = C_2 + C_3 \int_{|x|>1/t_0} (\log |x|)^{p-1} |x|^{\alpha} \nu_\rho(dx) < \infty
\end{aligned}
$$

by (6.13) and (6.10). Hence it follows from Proposition 3.18 (iii) that $\rho \in \mathfrak{D}^0(\Lambda_{p,\alpha})$. Thus $\mathfrak{D}^{\mathrm{e}}(\Lambda_{p,\alpha}) \subset \mathfrak{D}^0(\Lambda_{p,\alpha})$ and the assertion is true. In the case $\alpha = 0$, the argument is similar; it is done in Theorem 5.15 of [41] and Proposition 4.3 of [43].

(iv) Let $\alpha = 1$ and $p \ge 1$. Assume that $\rho \in \mathfrak{D}(\Lambda_{p,1})$. Then γ_{μ_t} given by

$$
\gamma_{\mu_t} = \int_0^t l(s)ds \left(\gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|l(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\, \nu_\rho(dx) \right)
$$

is convergent in $\mathbb{R}^d$ as $t \to \infty$ (Proposition 3.18). Since

$$
\int_{\mathbb{R}^d} x(1_{\{|l(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\, \nu_\rho(dx) \to \int_{|x|>1} x \nu_\rho(dx), \quad s \to \infty,
$$

and since $\int_0^\infty l(s)ds = \infty$ from (6.5), we have $\gamma_\rho = -\int_{|x|>1} x\nu_\rho(dx)$, that is, ρ has mean 0. Hence we have, with $\varepsilon = l(t)$,

$$
\begin{aligned}
\gamma_{\mu_t} &= \int_0^t l(s)ds \int_{\mathbb{R}^d} x(1_{\{|l(s)x|\le 1\}} - 1) \nu_\rho(dx) = -\int_0^t l(s)ds \int_{|l(s)x|>1} x\nu_\rho(dx) \\
&= -c_p \int_\varepsilon^1 (-\log t)^{p-1} t^{-1} dt \int_{|tx|>1} x\nu_\rho(dx) \\
&= -c_p \int_{|x|>1} x\nu_\rho(dx) \int_{\varepsilon\vee(1/|x|)} (-\log t)^{p-1} t^{-1} dt \\
&= -c_{p+1} \int_{|x|>1} x(\log(|x| \wedge (1/\varepsilon)))^p \nu_\rho(dx).
\end{aligned}
$$

Therefore ρ is in the right-hand side of (6.15). Conversely, if ρ is in the right-hand side of (6.15), then it follows from the equalities above that γ_{μ_t} is convergent, hence $\rho \in \mathfrak{D}(\Lambda_{p,1})$.

Assume that $\rho \in \mathfrak{D}^0(\Lambda_{p,1})$. Then $\rho \in \mathfrak{D}(\Lambda_{p,1})$, $\nu_\rho \in \mathfrak{D}(\Lambda_{p,1}^L)$, and ρ has mean 0. We have

$$\begin{aligned}\infty &> \int_0^\infty l(s)ds \left|\gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|l(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\,\nu_\rho(dx)\right| \\ &= \int_0^\infty l(s)ds \left|\int_{|l(s)x|>1} x\nu_\rho(dx)\right| = c_p \int_0^1 (-\log t)^{p-1} t^{-1} dt \left|\int_{|tx|>1} x\nu_\rho(dx)\right|,\end{aligned}$$

and ρ is in the right-hand side of (6.16). These equalities also show the converse.

(v) Let $1 < \alpha < 2$. Then $\int_0^\infty l(s)ds = \infty$. If $\rho \in \mathfrak{D}(\Lambda_{p,\alpha})$, then, by the same argument, $\gamma_\rho = -\int_{|x|>1} x\nu_\rho(dx)$, that is, ρ has mean 0. If $\rho \in ID$ has mean 0 and $\nu_\rho \in \mathfrak{D}(\Lambda_{p,\alpha}^L)$, then $\rho \in \mathfrak{D}^0(\Lambda_{p,\alpha})$, since

$$\begin{aligned}&\int_0^\infty l(s)ds \left|\gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|l(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\,\nu_\rho(dx)\right| \\ &\quad = \int_0^\infty l(s)ds \left|\int_{|l(s)x|>1} x\nu_\rho(dx)\right| = c_p \int_0^1 (-\log t)^{p-1} t^{-\alpha} dt \left|\int_{|tx|>1} x\nu_\rho(dx)\right| \\ &\quad \le c_p \int_{|x|>1} |x|\nu_\rho(dx) \int_{1/|x|}^1 (-\log t)^{p-1} t^{-\alpha} dt \\ &\quad \le C_4 \int_{|x|>1} (\log|x|)^{p-1} |x|^\alpha \nu_\rho(dx) < \infty\end{aligned}$$

from (6.14) and (6.10).

(vi) Let $\alpha \ge 2$. We have $\int_1^\infty l(s)^2 ds = \infty$ as well as $\int_1^\infty l(s)ds = \infty$ from (6.5). Combining this with (6.11), we obtain the result. $\square$

Remark 6.4. Open problem: Describe the domains of $\Lambda_{p,1}$ for $0 < p < 1$. $\square$

Remark 6.5. Consider the case where $\alpha = 1$ and $p \ge 1$. In this case,

$$\mathfrak{D}(\Lambda_{p,1}) \subsetneqq \{\rho \in ID\colon\, \nu_\rho \in \mathfrak{D}(\Lambda_{p,1}^L),\ \int_{\mathbb{R}^d} x\rho(dx) = 0\}. \tag{6.17}$$

Indeed, let $\lambda = \delta_{\xi_0}$, $\xi_0 \in S$, $q \in (p, p+1]$, and

$$\nu(B) = \int_S \lambda(d\xi) \int_2^\infty 1_B(r\xi) r^{-2} (\log r)^{-q} dr.$$

Then $\int_{|x|>2} |x|\nu(dx) < \infty$. Since $q > p$, $\nu \in \mathfrak{D}(\Lambda_{p,1}^L)$ by Theorem 6.2. Let $\rho \in ID$ be such that A_ρ arbitrary, $\nu_\rho = \nu$, and $\gamma_\rho = -\int_{|x|>1} x\nu(dx)$. Then ρ is in the right-hand side of (6.17), but $\rho \notin \mathfrak{D}(\Lambda_{p,1})$ by virtue of (6.15), because

$$\begin{aligned}&\int_{|x|>1} x(\log(|x| \wedge (1/\varepsilon)))^p \nu_\rho(dx) = p\int_\varepsilon^1 (-\log t)^{p-1} t^{-1} dt \int_{|x|>1/t} x\nu_\rho(dx) \\ &\quad = p\xi_0 \int_\varepsilon^1 (-\log t)^{p-1} t^{-1} dt \int_{(1/t)\vee 2}^\infty (\log r)^{-q} r^{-1} dr\end{aligned}$$

$$= \frac{p\xi_0}{q-1}\int_\varepsilon^1 (-\log t)^{p-1} t^{-1} (-\log(t\wedge(1/2)))^{1-q} dt,$$

which tends to the infinity point in the direction of ξ_0 as $\varepsilon \downarrow 0$, since $q \le p+1$. $\square$

6.2 Range of $\Lambda_{p,\alpha}^L$

Theorem 6.6. *Let $-\infty < \alpha < 2$ and $p > 0$.*

(i) *Let $\nu \in \mathfrak{D}(\Lambda_{p,\alpha}^L)$ with a radial decomposition $(\lambda(d\xi), \nu_\xi)$ and let $\widetilde{\nu} = \Lambda_{p,\alpha}^L \nu$. Then $\widetilde{\nu}$ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du)$, where*

$$h_\xi(y) = c_p \int_{(y,\infty)} (w-y)^{p-1} e^{\alpha w} \nu_\xi^\sharp(dw), \quad y \in \mathbb{R}, \tag{6.18}$$

$$\nu_\xi^\sharp(E) = \int_{(0,\infty)} 1_E(\log r) \nu_\xi(dr), \quad E \in \mathscr{B}(\mathbb{R}). \tag{6.19}$$

(ii) *$\Lambda_{p,\alpha}^L$ is one-to-one.*

Proof. (i) It follows from (6.12) that

$$\begin{aligned}
\widetilde{\nu}(B) &= c_p \int_S \lambda(d\xi) \int_{(0,\infty)} \nu_\xi(dr) \int_0^1 (-\log t)^{p-1} t^{-\alpha-1} 1_B(tr\xi) dt \\
&= c_p \int_S \lambda(d\xi) \int_{(0,\infty)} r^\alpha \nu_\xi(dr) \int_0^r (\log(r/u))^{p-1} u^{-\alpha-1} 1_B(u\xi) du \\
&= c_p \int_S \lambda(d\xi) \int_0^\infty u^{-\alpha-1} 1_B(u\xi) du \int_{(u,\infty)} (\log(r/u))^{p-1} r^\alpha \nu_\xi(dr) \\
&= \int_S \lambda(d\xi) \int_{(0,\infty)} 1_B(u\xi) u^{-\alpha-1} h_\xi(\log u) du,
\end{aligned}$$

where h_ξ is defined by (6.18) and (6.19).

(ii) Similarly to the proof of Theorem 4.9 (ii), we see that there is a measurable function $c(\xi)$ satisfying $0 < c(\xi) < \infty$, $c(\xi)\lambda'(d\xi) = \lambda(d\xi)$, and $u^{-\alpha-1} h'_\xi(\log u) du = c(\xi) u^{-\alpha-1} h_\xi(\log u) du$ on $\mathbb{R}_+^\circ$ for λ-a. e. ξ. Thus $h'_\xi(y)dy = c(\xi) h_\xi(y) dy$ on $\mathbb{R}$ for λ-a. e. ξ. For λ-a. e. ξ, $h_\xi(\log u)du$ and $h'_\xi(\log u)du$ are locally finite measures on $\mathbb{R}_+^\circ$, hence $h_\xi(y)dy$ and $h'_\xi(y)dy$ are locally finite measures on $\mathbb{R}$, and also $e^{\alpha w}\nu_\xi^\sharp(dw)$ and $e^{\alpha w}\nu_\xi'^\sharp(dw)$ are locally finite measures on $\mathbb{R}$. Now from Theorem 2.10 on the one-to-one property of I^p we obtain $e^{\alpha w}\nu_\xi'^\sharp(dw) = c(\xi) e^{\alpha w}\nu_\xi^\sharp(dw)$ for λ-a. e. ξ. Hence $\nu'_\xi = c(\xi)\nu_\xi$ for λ-a. e. ξ. This shows that $\nu' = \nu$. $\square$

Theorem 6.7. *Let $-\infty < \alpha < 2$ and $p > 0$. A measure η on $\mathbb{R}^d$ belongs to $\mathfrak{R}(\Lambda_{p,\alpha}^L)$ if and only if η is in $\mathfrak{M}^L$ and has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du)$ such that*

$$\begin{aligned}&h_\xi(y) \textit{ is measurable in } (\xi,y) \textit{ and, for } \lambda\textit{-a. e. } \xi,\\ &\textit{monotone of order } p \textit{ on } \mathbb{R} \textit{ in } y.\end{aligned} \tag{6.20}$$

Proof. This follows from Theorem 6.6. We can supply the details, modifying the proof of Theorem 4.10. □

Notice that monotonicity of order p on $\mathbb{R}_+^\circ$ and on $\mathbb{R}$ appears in (4.34) and in (6.20), respectively. This is why we have studied in Section 2 monotonicity on $\mathbb{R}_+^\circ$ and $\mathbb{R}$ both.

6.3 Classes $L_{p,\alpha}$, $L^0_{p,\alpha}$, and $L^{\mathrm{e}}_{p,\alpha}$

Define, for $-\infty < \alpha < 2$ and $p > 0$,

$$L_{p,\alpha} = L_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}(\Lambda_{p,\alpha}), \tag{6.21}$$

$$L^0_{p,\alpha} = L^0_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}^0(\Lambda_{p,\alpha}), \tag{6.22}$$

$$L^{\mathrm{e}}_{p,\alpha} = L^{\mathrm{e}}_{p,\alpha}(\mathbb{R}^d) = \mathfrak{R}^{\mathrm{e}}(\Lambda_{p,\alpha}). \tag{6.23}$$

The notation $L_{n,0}$ for positive integers n is already introduced in Section 1.2 as the classes of n times selfdecomposable distributions, but this is consistent with (6.21) for $p = n$ and $\alpha = 0$, because the known characterization of the Lévy measures of n times selfdecomposable distributions in Theorem 3.2 of Sato [37] coincides with the description of $L_{p,\alpha}$ in Theorem 6.12. Another proof is to use the expression (6.3) for $l_{n,0}$ and to recall the result mentioned in Section 1.2. A third proof is to use $\Lambda_{p+q,0} = \Lambda_{q,0}\Lambda_{p,0}$ to be shown in Theorem 7.3 (ii).

Proposition 6.8. *We have*

$$L^0_{p,\alpha} = L_{p,\alpha} = L^{\mathrm{e}}_{p,\alpha} \quad \textit{for } -\infty < \alpha < 1, \tag{6.24}$$

$$L^0_{p,1} \subset L_{p,1} \subset L^{\mathrm{e}}_{p,1}, \tag{6.25}$$

$$L^0_{p,\alpha} = L_{p,\alpha} \subset L^{\mathrm{e}}_{p,\alpha} \quad \textit{for } 1 < \alpha < 2. \tag{6.26}$$

Proof. This is parallel to Proposition 4.11. For $0 \le \alpha < 1$, (6.24) is proved as in Proposition 4.5, using Theorem 6.3 instead of Theorem 4.2. For $1 \le \alpha < 2$, use Theorem 6.3 and (3.28). □

Theorem 6.9. *Let $-\infty < \alpha < 2$ and $p > 0$. Then $\mu \in L^{\mathrm{e}}_{p,\alpha}$ if and only if $\mu \in ID$ and its Lévy measure ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1}h_\xi(\log u)du)$ satisfying* (6.20).

Proof. Use Proposition 3.27 and Theorem 6.7. □

Proposition 6.10. *Let $0 < \alpha < 2$, $p > 0$, and $\mu \in L^{\mathrm{e}}_{p,\alpha}$. Then*

$$\int_{\mathbb{R}^d} |x|^\beta \mu(dx) < \infty \quad \textit{for all } \beta \in (0,\alpha). \tag{6.27}$$

Proof. We have $\nu_\mu = \Lambda^L_{p,\alpha}\nu$ for some $\nu \in \mathfrak{D}(\Lambda^L_{p,\alpha})$ and

$$\begin{aligned}\int_{|x|>1} |x|^\beta \nu_\mu(dx) &= c_p \int_0^1 (-\log t)^{p-1} t^{-\alpha-1} dt \int_{|tx|>1} |tx|^\beta \nu(dx)\\ &= c_p \int_{|x|>1} |x|^\beta \nu(dx) \int_{1/|x|}^1 (-\log t)^{p-1} t^{\beta-\alpha-1} dt\\ &\le \text{const} \int_{|x|>1} (\log|x|)^{p-1} |x|^\alpha \nu(dx) < \infty\end{aligned}$$

by (6.13) and Theorem 6.2. □

Remark 6.11. Let $0<\alpha<2$ and $p>0$.

(i) There is $\mu \in L^{\mathrm{e}}_{p,\alpha}$ such that $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) = \infty$.

(ii) There is a non-Gaussian $\mu \in L^{\mathrm{e}}_{p,\alpha}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x|^{\alpha'} \mu(dx) < \infty$ for all $\alpha' > 0$.

Indeed, (i) is a consequence of Theorem 7.11 and Proposition 7.16 in the later section. To see (ii), choose $h(y) = (-y)^{p-1} 1_{(-\infty,0)}(y)$ and consider μ such that ν_μ has a radial decomposition $(\lambda, u^{-\alpha-1} h(\log u) du) = (\lambda, u^{-\alpha-1}(-\log u)^{p-1} 1_{(0,1)}(u)\, du)$ with a nonzero finite measure λ. □

We give characterization of $L_{p,\alpha}$ for $\alpha \ne 1$. Recall that $L_{p,\alpha} = L^0_{p,\alpha}$ if $\alpha \ne 1$.

Theorem 6.12. *Let $\mu \in ID$.*

(i) *Let $-\infty < \alpha < 1$ and $p>0$. Then $\mu \in L_{p,\alpha}$ if and only if ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du)$ satisfying* (6.20).

(ii) *Let $1<\alpha<2$ and $p>0$. Then $\mu \in L_{p,\alpha}$ if and only if ν_μ has a radial decomposition $(\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du)$ satisfying* (6.20) *and μ has mean* 0.

Proof. (i) Use Proposition 6.8 and Theorem 6.9.

(ii) Let $\mu \in L_{p,\alpha}$. Then $\mu \in L^{\mathrm{e}}_{p,\alpha}$ from (6.26), and Theorem 6.9 says that ν_μ has $(\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du)$ satisfying (6.20). We have $\mu = \Lambda_{p,\alpha}\rho$ for some $\rho \in \mathfrak{D}(\Lambda_{p,\alpha})$. Thus, by Theorems 6.2 and 6.3, $\int_{|x|>2} (\log|x|)^{p-1} |x|^\alpha \nu_\rho(dx) < \infty$ and $\int_{\mathbb{R}^d} x\rho(dx) = 0$. Hence $\gamma_\rho = -\int_{|x|>1} x\nu_\rho(dx)$. Let $l = l_{p,\alpha}$. It follows from Proposition 5.9 that

$$\int_0^\infty ds \int_{|l(s)x|>1} |l(s)x| \nu_\rho(dx) < \infty, \tag{6.28}$$

since this integral equals $\int_{|x|>1} |x| \nu_\mu(dx)$. It follows that

$$\gamma_\mu = -\int_0^\infty ds \int_{|l(s)x|>1} l(s)x \nu_\rho(dx), \tag{6.29}$$

and hence $\gamma_\mu = -\int_{|x|>1} x\nu_\mu(dx)$, that is, μ has mean 0.

Conversely, assume that ν_μ has the property stated and that μ has mean 0. Then by Theorem 6.9, $\mu \in L^{\mathrm{e}}_{p,\alpha}$ and $\nu_\mu \in \mathfrak{R}(\Lambda^L_{p,\alpha})$. Choose ν such that $\Lambda^L_{p,\alpha}\nu = \nu_\mu$.

Then (6.28) and (6.29) hold with ν in place of ν_ρ. Let $A = (\int_0^\infty l_{p,\alpha}(s)^2 ds)^{-1} A_\mu$ and $\gamma = -\int_{|x|>1} x\nu(dx)$. Then $\rho \in ID$ with triplet (A, ν, γ) belongs to $\mathfrak{D}(\Lambda_{p,\alpha})$ from Theorem 6.3 and we have $\Lambda_{p,\alpha}\rho = \mu$. $\square$

Remark 6.13. Open problem: Describe the classes $L_{p,1}(\mathbb{R}^d)$ and $L^0_{p,1}(\mathbb{R}^d)$ for $p > 0$. $\square$

Theorem 6.14. *Let $-\infty < \alpha < 2$ and $0 < p < \infty$. The mapping $\Lambda_{p,\alpha}$ is one-to-one.*

This is proved from Theorem 6.6 (ii) in the same way as Theorem 4.23.

Here is the continuity property of distributions in $L_{p,\alpha}$.

Theorem 6.15. (i) *Let μ be a nondegenerate distribution in $L^{\mathrm{e}}_{p,\alpha}$ with $0 \le \alpha < 2$ and $p > 0$. Then μ is absolutely continuous with respect to d-dimensional Lebesgue measure.*

(ii) *Let $\mu = \Lambda_{p,\alpha}\rho$ with $\alpha < 0$ and $p > 0$. Then ν_μ is a finite measure if and only if ν_ρ is a finite measure. In particular, for any $\alpha < 0$ and $p > 0$, $L_{p,\alpha}$ contains some compound Poisson distribution.*

Proof. This is proved by the same idea as Theorem 4.24. The key formulas are, for ν^0 and ρ similarly defined,

$$\int_0^\infty u^{-\alpha-1} h_\xi(\log u) du = c_p \int_{(0,\infty)} r^\alpha \nu^0_\xi(dr) \int_0^r u^{-\alpha-1} (\log(r/u))^{p-1} du = \infty$$

for $\alpha \ge 0$ and

$$\nu_\mu(\mathbb{R}^d) = (-\alpha)^{-p} \nu_\rho(\mathbb{R}^d)$$

for $\alpha < 0$. $\square$

6.4 Relation between $K_{p,\alpha}$ and $L_{p,\alpha}$

We have $K_{1,\alpha} = L_{1,\alpha}$, $K^0_{1,\alpha} = L^0_{1,\alpha}$, and $K^{\mathrm{e}}_{1,\alpha} = L^{\mathrm{e}}_{1,\alpha}$ for $-\infty < \alpha < 2$ and, in particular, $K_{1,0} = L_{1,0} = L$. See (1.18), (1.19), and (6.1).

Theorem 6.16. *Let n be an integer ≥ 2. Then*

$$K^{\mathrm{e}}_{n,\alpha} \underset{\neq}{\supsetneqq} L^{\mathrm{e}}_{n,\alpha} \quad \text{for } -\infty < \alpha < 2 \tag{6.30}$$

$$K_{n,\alpha} \underset{\neq}{\supsetneqq} L_{n,\alpha} \quad \text{for } \alpha \in (-\infty, 1) \cup (1, 2), \tag{6.31}$$

$$K^0_{n,\alpha} \underset{\neq}{\supsetneqq} L^0_{n,\alpha} \quad \text{for } \alpha \in (-\infty, 1) \cup (1, 2). \tag{6.32}$$

Proof. To see $K^{\mathrm{e}}_{n,\alpha} \supset L^{\mathrm{e}}_{n,\alpha}$, compare Theorems 4.15 and 6.9; we can see that it is enough to show that if $h(y)$ is a function monotone of order n on $\mathbb{R}$, then $h(\log u)$ is monotone of order n on $\mathbb{R}^\circ_+$. Let us prove this assertion. If $n = 1$, then the assertion is clear from Proposition 2.11 (i). Let $n \ge 2$ and assume that the assertion is true for

$n-1$ in place of n. Let $h(y)$ be monotone of order n on $\mathbb{R}$. Then $h(y) = \int_y^\infty \varphi(s)ds$ with φ monotone of order $n-1$ on $\mathbb{R}$,

$$h(\log u) = \int_{\log u}^\infty \varphi(s)ds = \int_u^\infty \varphi(\log t)t^{-1}dt,$$

and $\varphi(\log t)$ is monotone of order $n-1$ on $\mathbb{R}_+^\circ$. Since t^{-1} is completely monotone on $\mathbb{R}_+^\circ$, $\varphi(\log t)t^{-1}$ is monotone of order $n-1$ on $\mathbb{R}_+^\circ$ by Lemma 5.18. Thus $h(\log u)$ is monotone of order n on $\mathbb{R}_+^\circ$. Hence $K_{n,\alpha}^{\mathrm{e}} \supset L_{n,\alpha}^{\mathrm{e}}$.

Next, let us show that $K_{n,\alpha}^{\mathrm{e}} \setminus L_{n,\alpha}^{\mathrm{e}} \neq \emptyset$ for $n \geq 2$. Let

$$k(u) = (1-u)^{n-1} 1_{(0,1)}(u) = \int_{(u,\infty)} (s-u)^{n-1} \delta_1(ds),$$

which is monotone of order n on $\mathbb{R}_+^\circ$. Let

$$h(y) = k(e^y) = (1-e^y)^{n-1} 1_{(-\infty,0)}(y).$$

Then $h(y)$ is not monotone of order n on $\mathbb{R}$, since

$$h''(y) = (n-1)(n-2)(1-e^y)^{n-3}e^{2y} - (n-1)(1-e^y)^{n-2}e^y < 0$$

for y sufficiently close to $-\infty$. Hence, for any finite measure λ on S, the Lévy measure with radial decomposition $(\lambda, u^{-\alpha-1}k(u)du)$ belongs to $\mathfrak{R}(\bar{\Phi}_{n,\alpha}^L) \setminus \mathfrak{R}(\Lambda_{n,\alpha}^L)$.

For $\alpha \in (-\infty, 1)$, (6.31) and (6.32) follow from (6.30) by the equalities (4.38) and (6.24). For $\alpha \in (1,2)$, (6.31) and (6.32) follow from (6.30) by adding the condition of having mean 0 in Theorems 4.18 and 6.12. □

Remark 6.17. Open questions: (i) Is it true that $K_{n,1} \supsetneqq L_{n,1}$ and $K_{n,1}^0 \supsetneqq L_{n,1}^0$ for integers $n \geq 2$?
(ii) What is the relation between $K_{p,\alpha}$ and $L_{p,\alpha}$ for non-integer $p > 0$? □

7 One-parameter subfamilies of $\{L_{p,\alpha}\}$

7.1 $L_{p,\alpha}$*,* $L_{p,\alpha}^0$*, and* $L_{p,\alpha}^{\mathrm{e}}$ *for* $p \in (0,\infty)$ *with fixed* α

We give a basic relation.

Theorem 7.1. *Let* $-\infty < \alpha < 2$, $p > 0$, *and* $q > 0$. *Then*

$$\Lambda_{q,\alpha}^L \Lambda_{p,\alpha}^L = \Lambda_{p+q,\alpha}^L. \tag{7.1}$$

Proof. First note that a special case of (2.4) with $\sigma = \delta_0$ gives

$$c_p c_q \int_u^0 (-r)^{q-1}(r-u)^{p-1}dr = c_{p+q}(-u)^{p+q-1}, \qquad u < 0,$$

that is, for $0 < w < 1$,

$$c_p c_q \int_w^1 (-\log u)^{q-1}(-\log(w/u))^{p-1} u^{-1} du = c_{p+q}(-\log w)^{p+q-1}. \tag{7.2}$$

Given $\nu \in \mathfrak{M}^L(\mathbb{R}^d)$, let $\nu^{(j)}(\{0\}) = 0$, $j = 1, 2$, and

$$\nu^{(1)}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(l_{p,\alpha}(s)x)\nu(dx),$$
$$\nu^{(2)}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(l_{q,\alpha}(s)x)\nu^{(1)}(dx)$$

for $B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Then

$$\begin{aligned}
\nu^{(2)}(B) &= c_q \int_0^1 (-\log u)^{q-1} u^{-\alpha-1} du \int_{\mathbb{R}^d} 1_B(ux)\nu^{(1)}(dx) \\
&= c_q c_p \int_0^1 (-\log u)^{q-1} u^{-\alpha-1} du \int_0^1 (-\log t)^{p-1} t^{-\alpha-1} dt \int_{\mathbb{R}^d} 1_B(utx)\nu(dx) \\
&= c_q c_p \int_0^1 (-\log u)^{q-1} u^{-1} du \int_0^u (-\log(w/u))^{p-1} w^{-\alpha-1} dw \int_{\mathbb{R}^d} 1_B(wx)\nu(dx) \\
&= c_q c_p \int_{\mathbb{R}^d} \nu(dx) \int_0^1 1_B(wx) w^{-\alpha-1} dw \int_w^1 (-\log u)^{q-1}(-\log(w/u))^{p-1} u^{-1} du \\
&= c_{p+q} \int_0^1 (-\log w)^{p+q-1} w^{-\alpha-1} dw \int_{\mathbb{R}^d} 1_B(wx)\nu(dx),
\end{aligned}$$

using (7.2). Hence

$$\nu^{(2)} \in \mathfrak{M}^L \quad \Leftrightarrow \quad \nu \in \mathfrak{D}(\Lambda^L_{p+q,\alpha}).$$

On the other hand,

$$\nu^{(2)} \in \mathfrak{M}^L \quad \Leftrightarrow \quad \nu^{(1)} \in \mathfrak{D}(\Lambda^L_{q,\alpha}).$$

Hence

$$\nu \in \mathfrak{D}(\Lambda^L_{p+q,\alpha}) \quad \Leftrightarrow \quad \nu^{(1)} \in \mathfrak{D}(\Lambda^L_{q,\alpha}),\ \nu \in \mathfrak{D}(\Lambda^L_{p,\alpha}),\ \Lambda^L_{p,\alpha}\nu = \nu^{(1)}.$$

It follows that $\mathfrak{D}(\Lambda^L_{p+q,\alpha}) = \mathfrak{D}(\Lambda^L_{q,\alpha}\Lambda^L_{p,\alpha})$ and that, if $\nu \in \mathfrak{D}(\Lambda^L_{p+q,\alpha})$, then $\Lambda^L_{p+q,\alpha}\nu = \Lambda^L_{q,\alpha}\Lambda^L_{p,\alpha}\nu$. □

Corollary 7.2. *We have*

$$\mathfrak{R}(\Lambda^L_{p,\alpha}) \supset \mathfrak{R}(\Lambda^L_{p',\alpha}) \quad \textit{for } -\infty < \alpha < 2 \textit{ and } 0 < p < p'. \tag{7.3}$$

This corollary follows also from Theorem 6.7.

Theorem 7.3. *Let $-\infty < \alpha < 2$, $p > 0$, and $q > 0$.*
(i) *If $\rho \in \mathfrak{D}^0(\Lambda_{p+q,\alpha})$, then $\rho \in \mathfrak{D}^0(\Lambda_{p,\alpha})$, $\Lambda_{p,\alpha}\rho \in \mathfrak{D}^0(\Lambda_{q,\alpha})$, and*

$$\Lambda_{p+q,\alpha}\,\rho = \Lambda_{q,\alpha}\Lambda_{p,\alpha}\,\rho \tag{7.4}$$

(ii) *If* $\alpha \neq 1$, *then*

$$\Lambda_{p+q,\alpha} = \Lambda_{q,\alpha}\Lambda_{p,\alpha} \tag{7.5}$$

Proof. (i) Let $\rho \in \mathfrak{D}^0(\Lambda_{p+q,\alpha})$. As in the proof of Theorem 7.1,

$$\begin{aligned} &c_p c_q \int_0^1 (-\log u)^{q-1} u^{-\alpha-1} du \int_0^1 |C_\rho(tuz)|(1-t)^{p-1} t^{-\alpha-1} dt \\ &\quad = c_{p+q} \int_0^1 |C_\rho(wz)|(1-w)^{p+q-1} w^{-\alpha-1} dw, \end{aligned}$$

which is finite since $h \in \mathfrak{D}^0(\Lambda_{p+q,\alpha})$. Then, we can use Fubini's theorem and obtain

$$\begin{aligned} &c_p c_q \int_0^1 (-\log u)^{q-1} u^{-\alpha-1} du \int_0^1 C_\rho(tuz)(1-t)^{p-1} t^{-\alpha-1} dt \\ &\quad = c_{p+q} \int_0^1 C_\rho(wz)(1-w)^{p+q-1} w^{-\alpha-1} dw. \end{aligned}$$

We have $\rho \in \mathfrak{D}^0(\Lambda_{p,\alpha})$ from (6.4), and

$$\begin{aligned} &c_q \int_0^1 |C_{\Lambda_{p,\alpha}\rho}(uz)|(-\log u)^{q-1} u^{-\alpha-1} du \\ &\qquad \leq c_p c_q \int_0^1 (-\log u)^{q-1} u^{-\alpha-1} du \int_0^1 |C_\rho(tuz)|(1-t)^{p-1} t^{-\alpha-1} dt < \infty. \end{aligned}$$

Hence $\Lambda_{p,\alpha}\rho \in \mathfrak{D}^0(\Lambda_{q,\alpha})$ and (7.4) holds.

(ii) Let $\alpha \neq 1$. Then we have $\mathfrak{D}(\Lambda_{r,\alpha}) = \mathfrak{D}^0(\Lambda_{r,\alpha})$ for all $r > 0$ by Theorem 6.3. If $\rho \in \mathfrak{D}(\Lambda_{p+q,\alpha})$, then $\rho \in \mathfrak{D}(\Lambda_{q,\alpha}\Lambda_{p,\alpha})$ and $\Lambda_{q,\alpha}\Lambda_{p,\alpha}\rho = \Lambda_{p+q,\alpha}\rho$ by (i). It remains to show that $\mathfrak{D}(\Lambda_{q,\alpha}\Lambda_{p,\alpha}) \subset \mathfrak{D}(\Lambda_{p+q,\alpha})$. Let $\rho \in \mathfrak{D}(\Lambda_{q,\alpha}\Lambda_{p,\alpha})$. This means that $\rho \in \mathfrak{D}(\Lambda_{p,\alpha})$ and $\Lambda_{p,\alpha}\rho \in \mathfrak{D}(\Lambda_{q,\alpha})$. Hence $\nu_\rho \in \mathfrak{D}(\Lambda^L_{p,\alpha})$ and $\Lambda^L_{p,\alpha}\nu_\rho \in \mathfrak{D}(\Lambda^L_{q,\alpha})$. Hence we have $\nu_\rho \in \mathfrak{D}(\Lambda^L_{p+q,\alpha})$ from Theorem 7.1. Hence $\rho \in \mathfrak{D}^{\mathrm{e}}(\Lambda_{p+q,\alpha})$. Now, if $\alpha < 1$, then $\rho \in \mathfrak{D}(\Lambda_{p+q,\alpha})$ since $\mathfrak{D}^{\mathrm{e}}(\Lambda_{p+q,\alpha}) = \mathfrak{D}(\Lambda_{p+q,\alpha})$. If $\alpha > 1$, then $\int_{\mathbb{R}^d} x\rho(dx) = 0$ from $\rho \in \mathfrak{D}(\Lambda_{p,\alpha})$, using Theorem 6.3, and hence $\rho \in \mathfrak{D}(\Lambda_{p+q,\alpha})$. □

Corollary 7.4. *For any positive integer n and $\alpha \in (-\infty,1) \cup (1,2)$, we have*

$$\Lambda_{n,0} = \underbrace{\Phi \cdots \Phi}_{n} \quad \textit{and} \quad \Lambda_{n,\alpha} = \underbrace{\bar{\Phi}_{1,\alpha} \cdots \bar{\Phi}_{1,\alpha}}_{n},$$

where Φ is defined by (1.11).

Proof. Combine (7.5) with $\Lambda_{1,0} = \Phi$ and $\Lambda_{1,\alpha} = \bar{\Phi}_{1,\alpha}$. □

Remark 7.5. Open question: Is (7.5) true also for $\alpha = 1$?

Corollary 7.6. *For $-\infty < \alpha < 2$ and $0 < p < p'$*

$$L^0_{p,\alpha} \supset L^0_{p',\alpha} \quad \textit{and} \quad L^{\mathrm{e}}_{p,\alpha} \supset L^{\mathrm{e}}_{p',\alpha}. \tag{7.6}$$

Proof. Use Corollary 7.2 and Theorem 7.3 (i). $\square$

We can strengthen Corollary 7.6 as follows.

Theorem 7.7. *For $-\infty < \alpha < 2$ and $p > 0$*

$$L^0_{p,\alpha} \supsetneqq \bigcup_{p' \in (p,\infty)} L^0_{p',\alpha} \quad \textit{and} \quad L^{\mathrm{e}}_{p,\alpha} \supsetneqq \bigcup_{p' \in (p,\infty)} L^{\mathrm{e}}_{p',\alpha}. \tag{7.7}$$

Proof. Let λ be a nonzero finite measure on S and let $h(y) = (-y)^{p-1} 1_{(-\infty,0)}(y)$. Then $h(\log u) = (-\log u)^{p-1} 1_{(0,1)}(u)$. The measure ν of polar product type $(\lambda(d\xi), u^{-\alpha-1} h(\log u) du)$ belongs to $\mathfrak{R}(\Lambda^L_{p,\alpha}) \setminus \bigcup_{p'>p} \mathfrak{R}(\Lambda^L_{p',\alpha})$, since $h_\xi(y)$ is monotone of order p but not of order p' (Example 2.17 (a)). It follows that $L^{\mathrm{e}}_{p,\alpha} \setminus \bigcup_{p'>p} L^{\mathrm{e}}_{p',\alpha} \neq \emptyset$. If $\alpha < 1$, then this also says that $L^0_{p,\alpha} \setminus \bigcup_{p'>p} L^0_{p',\alpha} \neq \emptyset$. If $1 < \alpha < 2$, then let $\mu \in ID$ be such that $\nu_\mu = \nu$ and, recalling that $\int_{|x|>1} |x| \nu(dx) < \infty$, choose $\gamma_\mu = -\int_{|x|>1} x \nu(dx)$ to see that $\mu \in L^0_{p,\alpha} \setminus \bigcup_{p'>p} L^0_{p',\alpha}$ by Theorem 6.12. Assuming that $\alpha = 1$, let λ satisfy $\int_S \xi \lambda(d\xi) = 0$ and let $\rho \in ID$ be such that $\Lambda^L_{p,1} \nu_\rho = \nu$ and $\gamma_\rho = 0$. We consider the proof of Theorem 6.7 and see that ν_ρ can be chosen to be of polar product type with the same λ. Hence

$$\int_{\mathbb{R}^d} x (1_{\{|l_{p,1}(s)x| \le 1\}} - 1_{\{|x| \le 1\}}) \nu_\rho(dx) = 0,$$

which shows that $\rho \in \mathfrak{D}^0(\Lambda_{p,1})$ by Proposition 3.18. Thus $\mu = \Lambda_{p,1}\rho$ has $\nu_\mu = \nu$ and $\gamma_\mu = 0$ and belongs to $L^0_{p,1} \setminus \bigcup_{p'>p} L^0_{p',1}$. $\square$

Remark 7.8. Since $L_{p,\alpha} = L^0_{p,\alpha}$ for $\alpha \in (-\infty,1) \cup (1,2)$, $L_{p,\alpha}$ has the properties similar to Corollary 7.6 and Theorem 7.7 if $\alpha \neq 1$. Open question: Is it true that $L_{p,1} \supset L_{p',1}$ for $0 < p < p'$ and $L_{p,1} \supsetneqq \bigcup_{p'>p} L_{p',1}$ for $p > 0$? $\square$

If $\alpha \le 0$, then the class $L^0_{p,\alpha}$ is continuous for decreasing p in the following sense.

Theorem 7.9. *Let $-\infty < \alpha \le 0$ and $p > 0$. Then*

$$\bigcap_{q \in (0,p)} L^0_{q,\alpha} = L^0_{p,\alpha}. \tag{7.8}$$

Proof. Let $\mu \in \bigcap_{q \in (0,p)} L^0_{q,\alpha}$. It is enough to prove that $\mu \in L^0_{p,\alpha}$. Let $(\lambda(d\xi), u^{-\alpha-1} h_\xi(\log u) du)$ be a radial decomposition of ν_μ. For any $q \in (0,p)$ there is $\sigma^q_\xi \in \mathfrak{D}(I^q)$ such that

$$h_\xi(y) = c_q \int_{(y,\infty)} (s-y)^{q-1} \sigma^q_\xi(ds), \qquad y \in \mathbb{R}.$$

Fix ξ for the moment and omit the subscript ξ. For $-\infty < a < b < \infty$

$$\int_a^b h(y) dy = \int_a^b (I^q \sigma^q)(dy) \ge c_{q+1} \int_{(a,b]} (s-a)^q \sigma^q(ds),$$

as in the proof of Proposition 2.1. Hence

$$\int_{a-1}^{b} h(y)dy \ge c_{q+1}\int_{[a,b]}(s-a+1)^q\sigma^q(ds) \ge c_{q+1}\sigma^q([a,b]).$$

Hence by the diagonal method we can select a sequence $q_n \uparrow p$ such that σ^{q_n} converges vaguely to a locally finite measure σ^p on $\mathbb{R}$, that is,

$$\int_{\mathbb{R}} f(s)\sigma^{q_n}(ds) \to \int_{\mathbb{R}} f(s)\sigma^p(ds), \qquad n\to\infty$$

for any continuous function f on $\mathbb{R}$ with compact support. We claim that $\sigma^p \in \mathfrak{M}_\infty^p(\mathbb{R})$. We have, for $0<\beta<q<p$,

$$\begin{aligned}\infty &> \int_1^\infty u^{-\alpha-1}h(\log u)du = \int_0^\infty e^{-\alpha y}h(y)dy \ge \int_0^\infty h(y)dy\\ &= \int_0^\infty dy\, c_q\int_{(y,\infty)}(s-y)^{q-1}\sigma^q(ds) = \int_{(0,\infty)}\sigma^q(ds)c_q\int_0^s(s-y)^{q-1}dy\\ &= c_{q+1}\int_{(0,\infty)}s^q\sigma^q(ds) \ge c_{q+1}\int_{(1,\infty)}s^\beta\sigma^q(ds).\end{aligned}$$

Thus, for any continuous function $g(s)\le s^\beta$ with compact support in $(1,\infty)$,

$$\int_1^\infty u^{-\alpha-1}h(\log u)du \ge c_{q+1}\int_{(1,\infty)} g(s)\sigma^q(ds).$$

Letting $q=q_n$ and $n\to\infty$, we get the same inequality with σ^p in place of σ^q. Hence

$$\int_1^\infty u^{-\alpha-1}h(\log u)du \ge c_{p+1}\int_{(1,\infty)} s^\beta\sigma^p(ds).$$

Letting $\beta\uparrow p$, we can replace β in this inequality by p. This shows that $\sigma^p \in \mathfrak{M}_\infty^p(\mathbb{R})$. Next we claim that

$$h(y)dy = (I^p\sigma^p)(dy), \tag{7.9}$$

that is,

$$h(y) = c_p\int_{(y,\infty)}(s-y)^{p-1}\sigma^p(ds) \quad \text{for a. e. } y\in\mathbb{R}.$$

If this is shown, then we obtain $\mu\in L_{p,\alpha}$.

The proof of (7.9) is as follows. Let $\tau(dy)=h(y)dy$. Note that $\tau\in\mathfrak{D}(I^1)$. For large n we have $I^{p-q_n}\tau = I^{p-q_n}I^{q_n}\sigma^{q_n} = I^p\sigma^{q_n}$. As $n\to\infty$, $I^{p-q_n}\tau$ tends to τ vaguely on $\mathbb{R}$ by Lemma 2.9. So, it is enough to show that

$$I^p\sigma^{q_n} \to I^p\sigma^p \quad (\text{vaguely on } \mathbb{R}), \quad n\to\infty. \tag{7.10}$$

We write q for q_n. Let f be a continuous function on $\mathbb{R}$ with support in $[a,b]$. We have

$$
\begin{aligned}
\int f(r) I^p \sigma^q(dr) &= \int_{\mathbb{R}} f(r)dr \int_{(r,\infty)} c_p (s-r)^{p-1} \sigma^q(ds) \\
&= \int_{\mathbb{R}} \sigma^q(ds) \int_{-\infty}^{s} c_p (s-r)^{p-1} f(r)dr = \int_{\mathbb{R}} \sigma^q(ds) \int_0^\infty c_p u^{p-1} f(s-u)du \\
&= \int_{(-\infty,s_0]} \sigma^q(ds) \cdots + \int_{(s_0,\infty)} \sigma^q(ds) \cdots = J_1 + J_2.
\end{aligned}
$$

If s_0 is a continuity point of σ^p, then

$$
J_1 \to \int_{(-\infty,s_0]} \sigma^p(ds) \int_0^\infty c_p u^{p-1} f(s-u)du,
$$

since $\sigma^q \to \sigma^p$ vaguely on $\mathbb{R}$. Concerning J_2, we have

$$
\left| \int_0^\infty c_p u^{p-1} f(s-u)du \right| \le ||f|| c_p \int_{s-b}^{s-a} u^{p-1} du \sim ||f|| c_p (b-a) s^{p-1}, \quad s \to \infty.
$$

Let $0 < \varepsilon < p \wedge 1$. Let us show that

$$
\sup_{q \in (p-\varepsilon,p)} \int_{(c,\infty)} s^{p-1} \sigma^q(ds) \to 0, \qquad c \to \infty. \tag{7.11}
$$

Let $c > 1$. We have

$$
\begin{aligned}
\int_c^\infty h(y)dy &= c_{q+1} \int_{(c,\infty)} (s-c)^q \sigma^q(ds) \\
&\ge c_{p+1} \int_{(2c,\infty)} s^{p-\varepsilon} \frac{(s-c)^q}{s^{p-\varepsilon}} \sigma^q(ds) \ge c_{p+1} 2^{-p} \int_{(2c,\infty)} s^{p-\varepsilon} \sigma^q(ds),
\end{aligned}
$$

since, as $s \downarrow 2c$, $(s-c)^q / s^{p-\varepsilon} = (s-c)^{q-p+\varepsilon} (1-c/s)^{p-\varepsilon}$ decreases to $2^{-p+\varepsilon} c^{q-p+\varepsilon} \ge 2^{-p+\varepsilon} \ge 2^{-p}$. It follows that

$$
\int_c^\infty h(y)dy \ge c_{p+1} 2^{-p} \int_{(2c,\infty)} s^{p-1} \sigma^q(ds),
$$

which proves (7.11). Therefore J_2 is uniformly small if s_0 is close to ∞, and we obtain (7.10). □

Remark 7.10. Open question: Can one extend (7.8) to the case $0 < \alpha < 2$? □

As $L^0_{p,\alpha}$ and $L^{\mathrm{e}}_{p,\alpha}$ are decreasing with respect to p, we define, for $-\infty < \alpha < 2$,

$$
L^0_{\infty,\alpha} = L^0_{\infty,\alpha}(\mathbb{R}^d) = \bigcap_{p>0} L^0_{p,\alpha}, \tag{7.12}
$$

$$
L^{\mathrm{e}}_{\infty,\alpha} = L^{\mathrm{e}}_{\infty,\alpha}(\mathbb{R}^d) = \bigcap_{p>0} L^{\mathrm{e}}_{p,\alpha}. \tag{7.13}
$$

These are described by L_∞ and L^E_∞, $E \in \mathscr{B}((0,2))$, introduced in Section 1.4.

Theorem 7.11. *Descriptions of $L^{\mathrm{e}}_{\infty,\alpha}$ for all α and $L^{0}_{\infty,\alpha}$ for $\alpha \neq 1$ are as follows.*

$$L^{\mathrm{e}}_{\infty,\alpha} = L_{\infty} \qquad \text{for } -\infty < \alpha \le 0, \tag{7.14}$$

$$L^{\mathrm{e}}_{\infty,\alpha} = L^{(\alpha,2)}_{\infty} \qquad \text{for } 0 < \alpha < 2, \tag{7.15}$$

$$L^{0}_{\infty,\alpha} = L^{\mathrm{e}}_{\infty,\alpha} \qquad \text{for } -\infty < \alpha < 1, \tag{7.16}$$

$$L^{0}_{\infty,\alpha} = \{\mu \in L^{(\alpha,2)}_{\infty} \colon \textstyle\int_{\mathbb{R}^d} x\mu(dx) = 0\} \quad \text{for } 1 < \alpha < 2. \tag{7.17}$$

Proof. (7.14) and (7.15): First, notice that $L^{\mathrm{e}}_{\infty,\alpha} = \bigcap_{n=1,2,\ldots} L^{\mathrm{e}}_{n,\alpha}$. Let $\mu \in L^{\mathrm{e}}_{\infty,\alpha}$. Use Theorem 6.9. Then, for each $n = 1,2,\ldots,$ ν_μ has a radial decomposition $(\lambda^{(n)}(d\xi), u^{-\alpha-1}h^{(n)}_{\xi}(\log u)du)$, where $h^{(n)}_{\xi}(y)$ is measurable in (ξ,y) and, for $\lambda^{(n)}$-a. e. ξ, $h^{(n)}_{\xi}(y)$ is monotone of order n on $\mathbb{R}$. It follows from Proposition 3.1 that we can choose $\lambda^{(n)} = \lambda$ and $h^{(n)}_{\xi} = h_\xi$ independently of n. Thus, for λ-a. e. ξ, $h_\xi(y)$ is completely monotone on $\mathbb{R}$. We choose a modification of $h_\xi(y)$ completely monotone on $\mathbb{R}$ for all $\xi \in S$. Further, we choose λ to be a probability measure. For $y_0 \in \mathbb{R}$, the function $h_\xi(y_0+y)$, $y \in \mathbb{R}^{\circ}_{+}$, is completely monotone on $\mathbb{R}^{\circ}_{+}$ and hence

$$h_\xi(y_0+y) = \int_{(0,\infty)} e^{-y\beta}\Gamma^{y_0}_{\xi}(d\beta), \qquad y > 0$$

with a unique measure $\Gamma^{y_0}_{\xi}$ on $(0,\infty)$ by Bernstein's theorem (recall that our definition of complete monotonicity involves $h_\xi(y_0+y) \to 0$ as $y \to \infty$, so that $\Gamma^{y_0}_{\xi}$ has no mass at 0). In particular, we have Γ^{0}_{ξ} for $y_0 = 0$. If $y_0 < 0$, then

$$h_\xi(y) = h_\xi(y_0+(y-y_0)) = \int_{(0,\infty)} e^{-(y-y_0)\beta}\Gamma^{y_0}_{\xi}(d\beta), \quad y > 0$$

and hence $e^{y_0\beta}\Gamma^{y_0}_{\xi}(d\beta) = \Gamma^{0}_{\xi}(d\beta)$. Thus

$$h_\xi(y_0+y) = \int_{(0,\infty)} e^{-(y_0+y)\beta}\Gamma^{0}_{\xi}(d\beta), \quad y_0 < 0,\ y > 0.$$

Therefore

$$h_\xi(y) = \int_{(0,\infty)} e^{-y\beta}\Gamma^{0}_{\xi}(d\beta), \qquad y \in \mathbb{R}.$$

We see that $\{\Gamma^{0}_{\xi} \colon \xi \in S\}$ is a measurable family. Indeed, if Γ^{0}_{ξ} is a continuous measure for every ξ, then it is proved from the inversion formula (see [55], p. 285)

$$\int_0^s \Gamma^{0}_{\xi}(d\beta) = \lim_{y\to\infty} \sum_{m=0}^{[ys]} \frac{(-y)^m}{m!}(d/dy)^m(h_\xi(y)), \quad s > 0,$$

where $[ys]$ is the largest integer $\le ys$. If not, it is proved by approximating Γ^{0}_{ξ} by the convolutions with continuous measures. We have

$$\begin{aligned}
\infty > \int_{|x|\le 1} |x|^2 \nu_\mu(dx) &= \int_S \lambda(d\xi) \int_0^1 u^{1-\alpha} h_\xi(\log u) du \\
&= \int_S \lambda(d\xi) \int_0^1 u^{1-\alpha} du \int_{(0,\infty)} u^{-\beta} \Gamma_\xi^0(d\beta) \\
&= \int_S \lambda(d\xi) \int_0^1 u du \int_{(\alpha,\infty)} u^{-\beta} \Gamma_\xi(d\beta) \\
&= \int_S \lambda(d\xi) \int_{(\alpha,\infty)} \Gamma_\xi(d\beta) \int_0^1 u^{1-\beta} du,
\end{aligned}$$

where we define

$$\Gamma_\xi(E) = \int_{(0,\infty)} 1_E(\alpha+\beta) \Gamma_\xi^0(d\beta), \quad E \in \mathscr{B}((\alpha,\infty)).$$

Since $\int_0^1 u^{1-\beta} du = \infty$ for $\beta \ge 2$, we obtain $\Gamma_\xi([2,\infty)) = 0$ for λ-a. e. ξ. We have

$$\int_{|x|\le 1} |x|^2 \nu_\mu(dx) = \int_S \lambda(d\xi) \int_{(\alpha,2)} (2-\beta)^{-1} \Gamma_\xi(d\beta).$$

We also have

$$\begin{aligned}
\infty > \int_{|x|>1} \nu_\mu(dx) &= \int_S \lambda(d\xi) \int_1^\infty u^{-\alpha-1} h_\xi(\log u) du \\
&= \int_S \lambda(d\xi) \int_1^\infty u^{-\alpha-1} du \int_{(0,\infty)} u^{-\beta} \Gamma_\xi^0(d\beta) \\
&= \int_S \lambda(d\xi) \int_1^\infty u^{-1} du \int_{(\alpha,2)} u^{-\beta} \Gamma_\xi(d\beta) \\
&= \int_S \lambda(d\xi) \int_{(\alpha,2)} \Gamma_\xi(d\beta) \int_1^\infty u^{-\beta-1} du,
\end{aligned}$$

and $\int_1^\infty u^{-\beta-1} du = \infty$ for $\beta \le 0$. Hence, if $\alpha < 0$, then $\Gamma_\xi((\alpha,0]) = 0$ for λ-a. e. ξ. For any $\alpha < 2$ we have

$$\int_{|x|>1} \nu_\mu(dx) = \int_S \lambda(d\xi) \int_{(\alpha\vee 0,2)} \beta^{-1} \Gamma_\xi(d\beta).$$

Similarly, it follows from

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} h_\xi(\log u) du \tag{7.18}$$

that

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_{(\alpha\vee 0,2)} \Gamma_\xi(d\beta) \int_0^\infty 1_B(u\xi) u^{-\beta-1} du. \tag{7.19}$$

The measure $\lambda(d\xi)\Gamma_\xi(d\beta)$ on $S \times (\alpha \vee 0, 2)$ is written to $\Gamma(d\beta)\lambda_\beta(d\xi)$, where $\Gamma(d\beta)$ is a measure on $(\alpha \vee 0, 2)$ satisfying

$$\int_{(\alpha\vee 0,2)}(\beta^{-1}+(2-\beta)^{-1})\Gamma(d\beta)<\infty$$

and $\{\lambda_\beta\colon \beta\in(\alpha\vee 0,2)\}$ is a measurable family of probability measures on S. Therefore $L^{\mathrm{e}}_{\infty,\alpha}\subset L_\infty^{(\alpha\vee 0,2)}$.

Conversely, suppose that $\mu\in L_\infty^{(\alpha\vee 0,2)}$ with Lévy measure ν_μ satisfying (1.6). Then, defining $\lambda(d\xi)$ and $\Gamma_\xi(d\beta)$ in the converse direction and letting $h_\xi(y)=\int_{(\alpha\vee 0,2)}e^{-y(\beta-\alpha)}\Gamma_\xi(d\beta)$, we see (7.19) and then (7.18) with $h_\xi(y)$ completely monotone on $\mathbb{R}$. Hence $\mu\in L^{\mathrm{e}}_{\infty,\alpha}$. This completes the proof of (7.14) and (7.15).

Assertions (7.16) and (7.17) follow from Theorem 6.12 (i) and (ii), respectively. Note that if $\mu\in L_\infty^{(\alpha,2)}$ with $1<\alpha<2$, then

$$\int_{|x|>1}|x|\nu_\mu(dx)=\int_{(\alpha,2)}\Gamma(d\beta)\int_S\lambda_\beta(d\xi)\int_1^\infty r^{-\beta}dr=\int_{(\alpha,2)}(\beta-1)^{-1}\Gamma(d\beta)<\infty$$

and hence $\int_{\mathbb{R}^d}|x|\mu(dx)<\infty$. □

Remark 7.12. Open problem: Give the description of the class $L^0_{\infty,1}$. □

Remark 7.13. Open question: Does there exist a function $f(s)$, $s\geq 0$, such that $L^0_{\infty,\alpha}$ or $L^{\mathrm{e}}_{\infty,\alpha}$ is equal to $\mathfrak{R}(\Phi_f)$, $\mathfrak{R}^0(\Phi_f)$, or $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$? In particular, for $L_\infty=L^0_{\infty,0}=L^{\mathrm{e}}_{\infty,0}$, this is a long-standing question. □

Theorem 7.14. *We have*

$$K^{\mathrm{e}}_{\infty,\alpha}\supsetneqq L^{\mathrm{e}}_{\infty,\alpha}\quad \textit{for } -\infty<\alpha<2 \tag{7.20}$$

$$K^{0}_{\infty,\alpha}\supsetneqq L^{0}_{\infty,\alpha}\quad \textit{for } \alpha\in(-\infty,1)\cup(1,2). \tag{7.21}$$

Proof. We know that $\mu\in K^{\mathrm{e}}_{\infty,\alpha}$ if and only if ν_μ has radial decomposition $(\lambda,u^{-\alpha-1}k_\xi(u)du)$ with $k_\xi(u)$ completely monotone on $\mathbb{R}_+^\circ$. On the other hand, $\mu\in L^{\mathrm{e}}_{\infty,\alpha}$ if and only if ν_μ has radial decomposition $(\lambda,u^{-\alpha-1}h_\xi(\log u)du)$ with $h_\xi(y)$ completely monotone on $\mathbb{R}$. Since the complete monotonicity of $h_\xi(y)$ on $\mathbb{R}$ implies that of $h_\xi(\log u)$ on $\mathbb{R}_+^\circ$, we have $K^{\mathrm{e}}_{\infty,\alpha}\supset L^{\mathrm{e}}_{\infty,\alpha}$. To see the strictness of the inclusion, use the functions $h(y)=e^{-ce^y}$ and $k(u)=h(\log u)=e^{-cu}$ with $c>0$; $k(u)$ is completely monotone on $\mathbb{R}_+^\circ$ but $h(y)$ is not completely monotone on $\mathbb{R}$, since

$$h''(y)=-h(y)ce^u(1-ce^y)<0$$

for y close to $-\infty$. Hence (7.20) is true.

Assertion (7.21) for $\alpha\in(-\infty,1)$ is automatic from (7.20). For $\alpha\in(1,2)$, combine (7.20) with the condition of zero mean. □

When $\mu\in L_\infty$, let Γ_μ denote the measure Γ in the representation (1.6) of ν_μ. We give some moment properties of distributions in L_∞.

Proposition 7.15. *Let* $\mu\in L_\infty$. *Let* $0<\alpha<2$.

(i) *If* $\Gamma_\mu((0,\alpha])>0$, *then* $\int_{\mathbb{R}^d}|x|^\alpha\mu(dx)=\infty$.

(ii) *Suppose that* $\Gamma_\mu((0,\alpha])=0$. *Then,* $\int_{\mathbb{R}^d}|x|^\alpha\mu(dx)<\infty$ *if and only if* $\int_{(\alpha,2)}(\beta-\alpha)^{-1}\Gamma_\mu(d\beta)<\infty$.

Proof. Since λ_β in (1.6) satisfies $\lambda_\beta(S)=1$, we have

$$\int_{|x|>1}|x|^\alpha\nu_\mu(dx)=\int_{(0,2)}\Gamma_\mu(d\beta)\int_1^\infty r^{\alpha-\beta-1}dr.$$

Since $\int_1^\infty r^{\alpha-\beta-1}dr$ is infinite for $\beta\le\alpha$ and $(\beta-\alpha)^{-1}$ for $\beta>\alpha$, our assertions follow. □

Proposition 7.16. (i) *Let* $\mu\in L_\infty$ *and suppose that* μ *is not Gaussian (that is,* $\nu_\mu\neq 0$*). Let* α_0 *be the infimum of the support of* Γ_μ. *Then* $\alpha_0\in[0,2)$ *and* $\int_{\mathbb{R}^d}|x|^\alpha\mu(dx)=\infty$ *for* $\alpha\in(\alpha_0,2)$. *If* $\alpha_0>0$, *then* $\int_{\mathbb{R}^d}|x|^\alpha\mu(dx)<\infty$ *for* $\alpha\in(0,\alpha_0)$.

(ii) *Let* $0<\alpha<2$. *There exists* $\mu\in L_\infty^{(\alpha,2)}$ *such that* $\int_{\mathbb{R}^d}|x|^\alpha\mu(dx)=\infty$.

Proof. Assertion (i) follows from Proposition 7.15. To see (ii), choose $\alpha'\in(\alpha,2)$, let $\Gamma(d\beta)=1_{(\alpha,\alpha')}(\beta)d\beta$, and use Proposition 7.15 (ii). □

Remark 7.17. The identity (7.5) expresses the iteration of $\Lambda_{p,\alpha}$ for $\alpha\neq 1$. The iteration of a stochastic integral mapping Φ_f generates nested classes of their ranges. The description of their intersection is an interesting problem. See Maejima and Sato [27] and the references therein. □

7.2 $L_{p,\alpha}$, $L^0_{p,\alpha}$, and $L^{\mathrm{e}}_{p,\alpha}$ for $\alpha\in(-\infty,2)$ with fixed p

Little is known about the one-parameter families $\{L_{p,\alpha}\colon \alpha\in(-\infty,2)\}$, $\{L^0_{p,\alpha}\colon \alpha\in(-\infty,2)\}$, and $\{L^{\mathrm{e}}_{p,\alpha}\colon \alpha\in(-\infty,2)\}$ for fixed p.

Lemma 7.18. *Let n be a positive integer. If* $f(r)$ *is monotone of order n on* $\mathbb{R}$, *then, for any* $a>0$, $e^{-ar}f(r)$ *is monotone of order n on* $\mathbb{R}$.

Proof. This follows from Lemma 5.17, as e^{-ar} is completely monotone on $\mathbb{R}$. □

Theorem 7.19. *Let n be a positive integer. Then, for* $-\infty<\alpha<\alpha'<2$,

$$L_{n,\alpha}\supsetneqq L_{n,\alpha'},\quad L^0_{n,\alpha}\supsetneqq L^0_{n,\alpha'},\quad \textit{and}\quad L^{\mathrm{e}}_{n,\alpha}\supsetneqq L^{\mathrm{e}}_{n,\alpha'}. \tag{7.22}$$

Proof. Step 1. Let us prove that $L^{\mathrm{e}}_{n,\alpha}\supset L^{\mathrm{e}}_{n,\alpha'}$. Let $\mu\in L^{\mathrm{e}}_{n,\alpha'}$. Then ν_μ has radial decomposition $(\lambda(d\xi),u^{-\alpha'-1}h_\xi(\log u)du)$ with $h_\xi(y)$ monotone of order n on $\mathbb{R}$. Let

$$h^\flat_\xi(y)=e^{-(\alpha'-\alpha)y}h_\xi(y),$$

Then $h^\flat_\xi(y)$ is monotone of order n on $\mathbb{R}$ by the lemma above, and $u^{-\alpha'-1}h_\xi(\log u)=u^{-\alpha-1}h^\flat_\xi(\log u)$. Hence $\mu\in L^{\mathrm{e}}_{n,\alpha}$.

Step 2. Let us prove $L_{n,\alpha} \supset L_{n,\alpha'}$ and $L^0_{n,\alpha} \supset L^0_{n,\alpha'}$. If $\alpha < 1$, then these follow from Step 1. Suppose $\alpha = 1$ and let $\mu \in L_{n,\alpha'} = L^0_{n,\alpha'}$. Then, $\int_{\mathbb{R}^d} |x|\mu(dx) < \infty$ and $\int_{\mathbb{R}^d} x\mu(dx) = 0$ from Theorem 6.12 (ii). Since $\mu \in L^{\mathrm{e}}_{n,1}$ from Step 1, $\nu_\mu \in \mathfrak{R}(\Lambda^L_{n,1})$. Thus there is $\nu^0 \in \mathfrak{D}(\Lambda^L_{n,1})$ such that $\nu_\mu = \Lambda^L_{n,1}\nu^0$. We have $\int_{|x|>1} |x|\nu^0(dx) < \infty$ from Theorem 6.2. Since $\int_{|x|>1} |x|\nu_\mu(dx) < \infty$, we have

$$\int_0^\infty ds \int_{|l_{n,1}(s)x|>1} |l_{n,1}(s)x|\nu^0(dx) < \infty.$$

Moreover,

$$\gamma_\mu = -\int_{|x|>1} x\nu_\mu(dx) = -\int_0^\infty ds \int_{|l_{n,1}(s)x|>1} l_{n,1}(s)x\nu^0(dx).$$

Choose $\rho \in ID$ such that $\nu_\rho = \nu^0$, $A_\rho = \left(\int_0^\infty l_{n,1}(s)^2 ds\right)^{-1} A_\mu$, and $\gamma_\rho = -\int_{|x|>1} x \nu^0(dx)$. Then it follows from Proposition 3.18 that $\rho \in \mathfrak{D}^0(\Lambda_{n,1})$ and $\Lambda_{n,1}\rho = \mu$. Hence $\mu \in L^0_{n,1} \subset L_{n,1}$. Similarly, if $\alpha > 1$ and if $\mu \in L_{n,\alpha'} = L^0_{n,\alpha'}$, then $\mu \in L_{n,\alpha} = L^0_{n,\alpha}$.

Step 3. To show the strictness of the inclusion, let λ be a non-zero finite measure on S and let $h(y) = (-y)^{n-1} 1_{(-\infty,0)}(y)$, which is monotone of order n on $\mathbb{R}$ (Example 2.17 (a)). Then $(\lambda(d\xi), u^{-\alpha-1}h(\log u)du)$ is a radial decomposition of a Lévy measure ν, since $\int_0^1 u^{1-\alpha}h(\log u)du < \infty$. Let $\mu \in ID$ with $\nu_\mu = \nu$. Then $\mu \in L^{\mathrm{e}}_{n,\alpha}$ but $\mu \notin L^{\mathrm{e}}_{n,\alpha'}$, as is seen by an argument similar to the proof of Theorem 5.19. Indeed, we have

$$u^{-\alpha-1}h(\log u) = u^{-\alpha'-1}h^\sharp(\log u)$$

for

$$h^\sharp(y) = e^{(\alpha'-\alpha)y}(-y)^{n-1}1_{(-\infty,0)}(y),$$

which is not monotone of any order on $\mathbb{R}$ from Proposition 2.13 (iii). Strictness of the first and second inclusions in (7.22) is obtained from that of the third. □

Remark 7.20. Open question: Is (7.22) true for $p \in \mathbb{R}^\circ_+$ in place of n ? □

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