Description of limits of ranges of iterations of stochastic integral mappings of infinitely divisible distributions

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Abstract. For infinitely divisible distributions $\rho$ on $\mathbb{R}^d$ the stochastic integral mapping $\Phi f \rho$ is defined as the distribution of improper stochastic integral $\int_0^\infty f(s) dX_s^{(\rho)}$, where $f(s)$ is a non-random function and $\{X_s^{(\rho)}\}$ is a Lévy process on $\mathbb{R}^d$ with distribution $\rho$ at time 1. For three families of functions $f$ with parameters, the limits of the nested sequences of the ranges of the iterations $\Phi^n f$ are shown to be some subclasses, with explicit description, of the class $L_\infty$ of completely selfdecomposable distributions. In the critical case of parameter 1, the notion of weak mean 0 plays an important role. Examples of $f$ with different limits of the ranges of $\Phi^n f$ are also given.

1. Introduction

Let $ID = ID(\mathbb{R}^d)$ be the class of infinitely divisible distributions on $\mathbb{R}^d$, where $d$ is a fixed finite dimension. For a real-valued locally square-integrable function $f(s)$ on $\mathbb{R}_+ = [0, \infty)$, let

\[ \Phi f \rho = \mathcal{L}\left( \int_0^\infty f(s) dX_s^{(\rho)} \right), \]

the law of the improper stochastic integral $\int_0^\infty f(s) dX_s^{(\rho)}$ with respect to the Lévy process $\{X_s^{(\rho)}: s \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\rho)}) = \rho$. This integral is the limit in probability of $\int_0^t f(s) dX_s^{(\rho)}$ as $t \to \infty$. The domain of $\Phi f$, denoted by $\mathfrak{D}(\Phi f)$, is the class of $\rho \in ID$ such that this limit exists. The range of $\Phi f$ is denoted by $\mathfrak{R}(\Phi f)$. If $f(s) = 0$ for $s \in (s_0, \infty)$, then $\Phi f \rho = \mathcal{L}\left( \int_0^{s_0} f(s) dX_s^{(\rho)} \right)$ and $\mathfrak{D}(\Phi f) = ID$. For many choices of $f$, the description of $\mathfrak{R}(\Phi f)$ is known; they are quite diverse. A seminal example is $\mathfrak{R}(\Phi f) = L = L(\mathbb{R}^d)$, the class of selfdecomposable distributions on $\mathbb{R}^d$, for $f(s) = e^{-s}$ (Wolfe (1982), Sato (1999), Rocha-Arteaga and Sato (2003)). The iteration $\Phi^n f$ is defined by $\Phi^1 f = \Phi f$ and $\Phi^{n+1} f \rho = \Phi f(\Phi^n f \rho)$ with $\mathfrak{D}(\Phi^{n+1} f) = \{\rho \in \mathfrak{D}(\Phi f): \Phi^n f \rho \in \mathfrak{D}(\Phi f)\}$. Then

\[ ID \supset \mathfrak{R}(\Phi f) \supset \mathfrak{R}(\Phi^2 f) \supset \cdots. \]
We define the limit class

$$\mathcal{R}_\infty(\Phi_f) = \bigcap_{n=1}^{\infty} \mathcal{R}(\Phi_f^n).$$

If \( f(s) = e^{-s} \), then \( \mathcal{R}(\Phi_f^n) \) is the class of \( n \) times selfdecomposable distributions and \( \mathcal{R}_\infty(\Phi_f) \) is the class \( L_\infty \) of completely selfdecomposable distributions, which is the smallest class that is closed under convolution and weak convergence and contains all stable distributions on \( \mathbb{R}^d \). This sequence and the class \( L_\infty \) were introduced by Urbanik (1973) and studied by Sato (1980) and others. If \( f(s) = (1-s)1_{[0,1]}(s) \), then \( \mathcal{R}_\infty(\Phi_f) = L_\infty \), which was established by Jurek (2004) and Maejima and Sato (2009); in this case \( \mathcal{R}(\Phi_f) \) is the class of \( s \)-selfdecomposable distributions in the terminology of Jurek (1985). The paper of Maejima and Sato (2009) showed \( \mathcal{R}_\infty(\Phi_f) = L_\infty \) in many cases including (1) \( f(s) = (- \log s)1_{[0,1]}(s) \), (2) \( s = \int_{f(s)}^{\infty} u^{-1}e^{-u}du (0 < s < \infty) \), (3) \( s = \int_{f(s)}^{\infty} e^{-u^2}du (0 < s < s_0 = \sqrt{\pi}/2) \). The classes \( \mathcal{R}(\Phi_f) \) corresponding to (1)–(3) are Goldie–Steutel–Bondesson class \( B \), Thorin class \( T \) (see Barndorff-Nielsen et al. (2006)), and the class \( G \) of generalized type \( G \) distributions, respectively. These results pose the problem what classes other than \( L_\infty \) can appear as \( \mathcal{R}_\infty(\Phi_f) \) in general.

For \(-\infty < \alpha < 2, p > 0, \) and \( q > 0, \) we consider the three families of functions \( \tilde{f}_{p,\alpha}(s), l_{q,\alpha}(s), \) and \( f_\alpha(s) \) as in [S] (we refer to Sato (2010) as [S]). We define \( \tilde{\Phi}_{p,\alpha}, \) \( \tilde{\Lambda}_{q,\alpha}, \) and \( \tilde{\Psi}_\alpha \) to be the mappings \( \Phi_f \) with \( f(s) \) equal to these functions, respectively. In this paper we will prove the following theorem on the classes \( \mathcal{R}_\infty(\Phi_f) \) of those mappings. The case \( \alpha = 1 \) is delicate. There the notion of weak mean 0 plays an important role.

**Theorem 1.1.** (i) If \( \alpha \leq 0, p \geq 1, \) and \( q > 0, \) then

$$\mathcal{R}_\infty(\tilde{\Phi}_{p,\alpha}) = \mathcal{R}_\infty(\tilde{\Lambda}_{q,\alpha}) = \mathcal{R}_\infty(\tilde{\Psi}_\alpha) = L_\infty.$$

(ii) If \( 0 < \alpha < 1, p \geq 1, \) and \( q > 0, \) then

$$\mathcal{R}_\infty(\tilde{\Phi}_{p,\alpha}) = \mathcal{R}_\infty(\tilde{\Lambda}_{q,\alpha}) = \mathcal{R}_\infty(\tilde{\Psi}_\alpha) = L_\infty^{(\alpha,2)}.$$

(iii) If \( \alpha = 1, p \geq 1, \) and \( q = 1, \) then

$$\mathcal{R}_\infty(\tilde{\Phi}_{p,1}) = \mathcal{R}_\infty(\tilde{\Lambda}_{1,1}) = \mathcal{R}_\infty(\tilde{\Psi}_1) = L_\infty^{(1,2)} \cap \{ \mu \in ID: \mu \text{ has weak mean } 0 \}.$$

(iv) If \( 1 < \alpha < 2, p \geq 1, \) and \( q > 0, \) then

$$\mathcal{R}_\infty(\tilde{\Phi}_{p,\alpha}) = \mathcal{R}_\infty(\tilde{\Lambda}_{q,\alpha}) = \mathcal{R}_\infty(\tilde{\Psi}_\alpha) = L_\infty^{(\alpha,2)} \cap \{ \mu \in ID: \mu \text{ has mean } 0 \}. $$
Let us explain the concepts used in the statement of Theorem 1.1. A distribution 
μ ∈ ID belongs to \( L_∞ \) if and only if its Lévy measure \( ν_μ \) is represented as
\[
ν_μ(B) = \int_{(0,2)} \Gamma_μ(dβ) \int_S \lambda_β^μ(dξ) \int_0^∞ 1_B(rξ)r^{-β-1}dr
\]
for Borel sets \( B \) in \( \mathbb{R}^d \), where \( \Gamma_μ \) is a measure on the open interval \((0,2)\) satisfying
\[
\int_{(0,2)}(β^{-1} + (2 - β)^{-1})Γ_μ(dβ) < ∞
\]
and \( \{λ_β^μ: β ∈ (0,2)\} \) is a measurable family of probability measures on \( S = \{ξ ∈ \mathbb{R}^d: |ξ| = 1\} \). This \( Γ_μ \) is uniquely determined by \( ν_μ \) and \( \{λ_β^μ\} \) is determined by \( ν_μ \) up to \( β \) of \( Γ_μ \)-measure 0 (see [S] and Sato (1980)). For a Borel subset \( E \) of the interval \((0,2)\), the class \( L^E_∞ \) denotes, as in [S], the totality of \( μ ∈ L_∞ \) such that \( Γ_μ \) is concentrated on \( E \). The classes \( L^{(α,2)}_∞ \) and \( L^{(1,2)}_∞ \) appearing in Theorem 1.1 are for \( E = (α,2) \) and \((1,2)\), respectively. Let \( C_μ(z) \) \((z ∈ \mathbb{R}^d)\), \( A_μ \), and \( ν_μ \) be the cumulant function, the Gaussian covariance matrix, and the Lévy measure of \( μ ∈ ID \). A distribution \( μ ∈ ID \) is said to have weak mean \( m_μ \) if \( \lim_{a→∞} \int_{|z|≤a} xν_μ(dx) \) exists in \( \mathbb{R}^d \) and if
\[
C_μ(z) = -\frac{1}{2}⟨z, A_μz⟩ + \lim_{a→∞} \int_{|z|≤a} (e^{i(z,x)} - 1 - i(z,x))ν_μ(dx) + i⟨m_μ, z⟩.
\]
This concept was introduced by [S] recently. If \( μ ∈ ID \) has mean \( m_μ \) (that is, \( \int_{\mathbb{R}^d} |x|μ(dx) < ∞ \) and \( \int_{\mathbb{R}^d} xμ(dx) = m_μ \)), then \( μ \) has weak mean \( m_μ \) (Remark 3.8 of [S]).

Section 2 begins with exact definitions of \( f_α \), \( \tilde{f}_{p,α} \), and \( l_{q,α} \) and expounds existing results concerning \( \mathcal{R}_∞(Φ_f) \). Then, in Section 3, we will prove Theorem 1.1. In Section 4 we will give examples of \( Φ_f \) for which \( \mathcal{R}_∞(Φ_f) \) is different from those appearing in Theorem 1.1. Section 5 gives some concluding remarks.

2. Known results

Let \(-∞ < α < 2 \), \( p > 0 \), and \( q > 0 \) and let
\[
\tilde{g}_{p,α}(t) = \frac{1}{Γ(p)} \int_t^1 (1 - u)^{p-1}u^{-α-1}du, \quad 0 < t ≤ 1,
\]
\[
j_{q,α}(t) = \frac{1}{Γ(q)} \int_t^1 (-\log u)^{q-1}u^{-α-1}du, \quad 0 < t ≤ 1,
\]
\[
g_α(t) = \int_t^∞ u^{-α-1}e^{-u}du, \quad 0 < t ≤ ∞.
\]
Let \( t = \tilde{f}_{p,α}(s) \) for \( 0 ≤ s ≤ \tilde{g}_{p,α}(0+) \), \( t = l_{q,α}(s) \) for \( 0 ≤ s ≤ j_{q,α}(0+) \), and \( t = f_α(s) \) for \( 0 ≤ s ≤ g_α(0+) \) be the inverse functions of \( s = \tilde{g}_{p,α}(t) \), \( s = j_{q,α}(t) \), and \( s = g_α(t) \), respectively. They are continuous, strictly decreasing functions. If \( α < 0 \), then
$g_{p,\alpha}(0+)$, $j_{q,\alpha}(0+)$, and $g_{\alpha}(0+)$ are finite and we define $\tilde{f}_{p,\alpha}(s)$, $l_{q,\alpha}(s)$, and $f_{\alpha}(s)$ to be zero for $s \geq g_{p,\alpha}(0+)$, $j_{q,\alpha}(0+)$, and $g_{\alpha}(0+)$, respectively. Let $\Phi_{p,\alpha}$, $L_{q,\alpha}$, and $\Psi_{\alpha}$ denote $\Phi_f$ with $f = \tilde{f}_{p,\alpha}$, $l_{q,\alpha}$, and $f_{\alpha}$, respectively. Let $K_{p,\alpha}$, $L_{q,\alpha}$, and $K_{\infty,\alpha}$ be the ranges of $\Phi_{p,\alpha}$, $L_{q,\alpha}$, and $\Psi_{\alpha}$, respectively. These mappings and classes were systematically studied in Sato (2006) and [S]. In the following cases we have explicit expressions:

\[
\begin{align*}
\tilde{f}_{1,\alpha}(s) &= l_{1,\alpha}(s) = \begin{cases} 
(1 - |\alpha|s)^{1/|\alpha|}1_{[0,1/|\alpha|]}(s) & \text{for } \alpha < 0, \\
 e^{-s} & \text{for } \alpha = 0, \\
(1 + \alpha s)^{-1/\alpha} & \text{for } \alpha > 0,
\end{cases} \\
\tilde{f}_{p,\infty}(s) &= \{1 - (\Gamma(p + 1)s)^{1/p}\}1_{[0,1/\Gamma(p+1)]}(s), \quad p > 0, \\
l_{q,0}(s) &= \exp(-(\Gamma(q+1)s)^{1/q}), \quad q > 0, \\
f_{-1}(s) &= (-\log s)1_{[0,1]}(s).
\end{align*}
\]

In the case $p = q = 1$ we have $\Phi_{1,\alpha} = L_{1,\alpha}$ and $K_{1,\alpha} = L_{1,\alpha}$, which are in essence treated earlier by Jurek (1988, 1989); $\Phi_{1,\alpha}$ were studied by Maejima et al. (2010a), and Maejima and Ueda (2010b) with the notation $\Phi_{\alpha}$. The mapping $L_{q,0}$ and the class $L_{q,0}$ with $q = 1, 2, \ldots$ coincide with those introduced by Jurek (1983) in a different form. A variant of $\Psi_{\alpha}$ is found in Grigelionis (2007).

A related family is

\[
G_{\alpha,\beta}(t) = \int_t^\infty u^{-\alpha-1}e^{-\beta u}du, \quad 0 < t \leq \infty,
\]

for $-\infty < \alpha < 2$ and $\beta > 0$. Let $t = G_{\alpha,\beta}^*(s)$ for $0 \leq s < G_{\alpha,\beta}(0+)$ be the inverse function of $s = G_{\alpha,\beta}(t)$. If $\alpha < 0$, then $G_{\alpha,\beta}(0+)$ is finite and we define $G_{\alpha,\beta}^*(s) = 0$ for $s \geq G_{\alpha,\beta}(0+)$. Let $\Psi_{\alpha,\beta}$ denote $\Phi_f$ with $f = G_{\alpha,\beta}^*$. This was introduced by Maejima and Nakahara (2009) and studied by Maejima and Ueda (2010b) and, in the level of Lévy measures, by Maejima et al. (2010c). Clearly, $\Psi_{\alpha,1} = \Psi_{\alpha}$. We have

\[
G_{-\beta,\beta}^*(s) = (-\log \beta s)^{1/\beta}1_{[0,1/\beta]}(s), \quad \beta > 0.
\]

Earlier the mappings $\Psi_{0,2}$ and $\Psi_{-\beta,\beta}$ were treated in Aoyama et al. (2008) and Aoyama et al. (2010), respectively; $\Psi_{-2,2}$ appeared also in Arizmendi et al. (2010).

Maejima and Sato (2009) proved the following two results.

**Proposition 2.1.** Let $0 < t_0 \leq \infty$. Let $h(u)$ be a positive decreasing function on $(0,t_0)$ such that $\int_0^{t_0}(1 + u^2)h(u)du < \infty$. Let $g(t) = \int_t^{t_0}h(u)du$ for $0 < t \leq t_0$. Let $t = f(s)$, $0 \leq s < g(0+)$, be the inverse function of $s = g(t)$ and let $f(s) = 0$ for $s \geq g(0+)$. Then $\mathfrak{R}_\infty(\Phi_f) = L_\infty$. 


Proposition 2.2. $R_\infty(\Psi_0) = L_\infty$.

It follows from Proposition 2.1 that $R_\infty(\Phi_f) = L_\infty$ for $f = \bar{\Phi}_{p,\alpha}$ with $p \geq 1$ and $-1 \leq \alpha < 0$, $f = l_{q, \alpha}$ with $q \geq 1$ and $-1 \leq \alpha < 0$, $f = f_\alpha$ with $-1 \leq \alpha < 0$, and $f = G_{\alpha, \beta}^*$ with $-1 \leq \alpha < 0$ and $\beta > 0$. The function $f_0$ for $\Psi_0 = \Phi_{f_0}$ does not satisfy the condition in Proposition 2.1 but Proposition 2.2 is proved using the identity $\Psi_0 = \Lambda_{1,0} \Psi_{-1} = \Psi_{-1} \Lambda_{1,0}$.

In November 2007–January 2008, Sato wrote four memos, showing the part related to $\Psi_\alpha$ in (ii), (iii), and (iv) of Theorem 1.1. But assertion (iii) for $\Psi_1$ was shown with the set $\{ \mu \in ID : \mu \text{ has weak mean } 0 \}$ replaced by the set of $\mu \in L_\infty$ satisfying some condition related to (4.6) of Sato (2006). At that time the concept of weak mean was not yet introduced. Those memos showed that some proper subclasses of $L_\infty$ appear as limit classes $R_\infty(\Phi_f)$.

Sato’s memos were referred to by a series of papers Maejima and Ueda (2009a, b, 2010a, b) and Ichifuji et al. (2010). In Maejima and Ueda (2010a, c) they characterized $R(\Lambda_{1, \alpha}^n)$, $-\infty < \alpha < 2$, for $n = 1, 2, \ldots$, in relation to a decomposability which they called $\alpha$-selfdecomposability, and found $R_\infty(\Lambda_{1, \alpha})$ for $-\infty < \alpha < 2$. But the description of $R_\infty(\Lambda_{1,1})$ was similar to Sato’s memos. In Maejima and Ueda (2010b) they showed that $\Psi_{\alpha, \beta}$ with $-\infty < \alpha < 2$ and $\beta > 0$ satisfies $R_\infty(\Psi_{\alpha, \beta}) = R_\infty(\Psi_\alpha)$, under the condition that $\alpha \neq 1 + n\beta$ for $n = 0, 1, 2, \ldots$. For $\Psi_{0,2}$ and $\Psi_{-1, \beta}$ with $\beta > 0$, this result was earlier obtained by Aoyama et al. (2010). Further it was shown in Maejima and Ueda (2009b) that $R_\infty(\Psi_\alpha) = R_\infty(\Lambda_{1, \alpha})$ for $-\infty < \alpha < 2$. An application of the result in Maejima and Ueda (2010a) was given in Ichifuji et al. (2010).

If $f(s) = b_1[0, a](s)$ for some $a > 0$ and $b \neq 0$, then it is clear that $R_\infty(\Phi_f) = R(\Phi_f) = ID$. A first example of $R_\infty(\Phi_f)$ satisfying $L_\infty \subsetneq R_\infty(\Phi_f) \subsetneq ID$ was given by Maejima and Ueda (2009a); they showed that if $f(s) = b^{-[s]}$ for a given $b > 1$ with $[s]$ being the largest integer not exceeding $s$, then $R_\infty(\Phi_f) = L_\infty(b)$, the smallest class that is closed under convolution and weak convergence and contains all semi-stable distributions on $\mathbb{R}^d$ with $b$ as a span; in this case $R(\Phi_f)$ is the class $L(b)$ of semi-selfdecomposable distributions on $\mathbb{R}^d$ with $b$ as a span. See Sato (1999) for the definitions of semi-stability, semi-selfdecomposability, and span. See Maejima et al. (2000) for characterization of $L_\infty(b)$ as the limit of the class $L_n(b)$ of $n$ times $b$-semi-selfdecomposable distributions and for description of the Lévy measures of distributions in $L_\infty(b)$. Recall that $L_\infty \subsetneq L_\infty(b)$.
We have the following result in [S].

**Proposition 2.3.** The assertions related to \( \Lambda_{q,\alpha} \) in (i), (ii), and (iv) of Theorem 1.1 are true.

Indeed, in [S], Theorem 7.3 says that \( \Lambda_{q+q',\alpha} = \Lambda_{q',\alpha} \Lambda_{q,\alpha} \) for \( \alpha \in (-\infty, 1) \cup (1, 2) \), \( q > 0 \), and \( q' > 0 \), and hence \( \Lambda^n_{q,\alpha} = \Lambda_{nq,\alpha} \), and further, Theorem 7.11 combined with Proposition 6.8 describes \( \bigcap_{q>0} L_{q,\alpha} \) for \( \alpha \in (-\infty, 1) \cup (1, 2) \).

3. Proof of Theorem 1.1

We prepare some lemmas. We use the terminology in [S] such as radial decomposition, monotonicity of order \( p \), and complete monotonicity. In particular, our complete monotonicity implies vanishing at infinity. The location parameter \( \gamma_\mu \) of \( \mu \in ID \) is defined by

\[
C_\mu(z) = -\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle 1_{\{|x| \leq 1\}}(x) \nu_\mu(dx) + i \langle \gamma_\mu, z \rangle.
\]

Let \( K^c_{p,\alpha} \) [resp. \( K^c_{\infty,\alpha} \)] denote the class of distributions \( \mu \in ID \) for which there exist \( \rho \in ID \) and a function \( q_t \) from \([0, \infty) \) into \( \mathbb{R}^d \) such that \( \int_0^t f_p(s) dX_s^{(\rho)} - q_t \) [resp. \( \int_0^t f_\alpha(s) dX_s^{(\rho)} - q_t \)] converges in probability as \( t \to \infty \) and the limit has distribution \( \mu \).

**Lemma 3.1.** Let \(-\infty < \alpha < 2\) and \( p > 0 \). The domains of \( \bar{\Phi}_{p,\alpha} \) and \( \Psi_\alpha \) are as follows:

\[
\mathcal{D}(\bar{\Phi}_{p,\alpha}) = \mathcal{D}(\Psi_\alpha) = \begin{cases} 
ID & \text{for } \alpha < 0, \\
\{ \rho \in ID : \int_{|x|>1} \log |x| \nu_\rho(dx) < \infty \} & \text{for } \alpha = 0, \\
\{ \rho \in ID : \int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty \} & \text{for } 0 < \alpha < 1, \\
\{ \rho \in ID : \int_{|x|>1} |x| \nu_\rho(dx) < \infty, \int_{\mathbb{R}^d} x \rho(dx) = 0, \\ 
\quad \lim_{s \to \infty} \int_{|x|>s} s^{-1} ds \int_{|x|>s} x \nu_\rho(dx) \text{ exists in } \mathbb{R}^d \} & \text{for } \alpha = 1, \\
\{ \rho \in ID : \int_{|x|>1} |x|^\alpha \nu_\rho(dx) < \infty, \int_{\mathbb{R}^d} x \rho(dx) = 0 \} & \text{for } 1 < \alpha < 2. 
\end{cases}
\]

This is found in Sato (2006) or Theorems 4.2, 4.4 and Propositions 4.6, 5.1 of [S].

**Lemma 3.2.** Let \(-\infty < \alpha < 2\) and \( p > 0 \). The class \( K^c_{p,\alpha} \) [resp. \( K^c_{\infty,\alpha} \)] is the totality of \( \mu \in ID \) for which \( \nu_\mu \) has a radial decomposition \( (\lambda_\mu(d\xi), u^{-\alpha-1} k^\alpha_\xi(u) du) \) such that \( k^\alpha_\xi(u) \) is measurable in \( (\xi, u) \) and, for \( \lambda_\mu \)-a. e. \( \xi \), monotone of order \( p \) [resp. completely
monotone] on \( \mathbb{R}^+ = (0, \infty) \) in \( u \). The classes \( K_{p,\alpha} \) and \( K_{\infty,\alpha} \), that is, the ranges of \( \Phi_{p,\alpha} \) and \( \Psi_{\alpha} \), are as follows:

\[
K_{p,\alpha} = \begin{cases} 
K^e_{p,\alpha} & \text{for } -\infty < \alpha < 1, \\
\{ \mu \in K^e_{p,1} : \mu \text{ has weak mean 0} \} & \text{for } \alpha = 1, \\
\{ \mu \in K^e_{p,\alpha} : \mu \text{ has mean 0} \} & \text{for } 1 < \alpha < 2,
\end{cases}
\]

\[
K_{\infty,\alpha} = \begin{cases} 
K^e_{\infty,\alpha} & \text{for } -\infty < \alpha < 1, \\
\{ \mu \in K^e_{\infty,1} : \mu \text{ has weak mean 0} \} & \text{for } \alpha = 1, \\
\{ \mu \in K^e_{\infty,\alpha} : \mu \text{ has mean 0} \} & \text{for } 1 < \alpha < 2.
\end{cases}
\]

See Theorems 4.18, 5.8, and 5.10 of [S]. Note that if \( \mu \) is in \( K^e_{\infty,\alpha} \) or \( K^e_{p,\alpha} \) with \( 0 < \alpha < 2 \), then \( \int_{\mathbb{R}^d} |x|^\beta \mu(dx) < \infty \) for \( \beta \in (0, \alpha) \) (Propositions 4.16 and 5.13 of [S]). It follows from the lemma above that \( K^e_{p,\alpha} \supset K^e_{p',\alpha} \) and \( K_{p,\alpha} \supset K_{p',\alpha} \) for \( p < p' \) and that \( K^e_{\infty,\alpha} = \bigcap_{p>0} K^e_{p,\alpha} \) and \( K_{\infty,\alpha} = \bigcap_{p>0} K_{p,\alpha} \). In fact, this is the reason why we use the notation \( K^e_{\infty,\alpha} \) and \( K_{\infty,\alpha} \).

**Lemma 3.3.** Let \( \rho \in L_\infty \).

(i) Let \( 0 < \alpha < 2 \). Then \( \int_{\mathbb{R}^d} |x|^\alpha \rho(dx) < \infty \) if and only if \( \Gamma_\rho((0, \alpha)) = 0 \) and \( \int_{(\alpha, 2)} (\beta - \alpha)^{-1} \Gamma_\rho(d\beta) < \infty \).

(ii) \( \int_{|x|>1} \log |x| \rho(dx) < \infty \) if and only if \( \int_{(0, 2)} \beta^{-2} \Gamma_\rho(d\beta) < \infty \).

**Proof.** Assertion (i) is shown in Proposition 7.15 of [S]. Since

\[
\int_{|x|>1} \log |x| \nu_\rho(dx) = \int_{(0, 2)} \Gamma_\rho(d\beta) \int_{\mathbb{R}^d} \int_1^\infty (\log |r\xi|) r^{-\beta-1} dr d\xi
\]

\[
= \int_{(0, 2)} \Gamma_\rho(d\beta) \int_1^\infty (\log r) r^{-\beta-1} dr = \int_{(0, 2)} \beta^{-2} \Gamma_\rho(d\beta),
\]

assertion (ii) follows. \( \square \)

**Lemma 3.4.** Let \( \mu \) and \( \rho \) be in \( L_{(1,2)} \). Suppose that \( \Gamma_\rho(d\beta) = (\beta - 1)b(\beta)\Gamma_\rho(d\beta) \) and \( \lambda_\beta = \lambda_\beta^\mu \) with a nonnegative measurable function \( b(\beta) \) such that \( (\beta - 1)^{-1} b(\beta) - 1 \) is bounded on \((1, 2)\). Then, \( \int_1^a s^{-1} ds \int_{|x|>s} x \nu_\rho(dx) \) is convergent in \( \mathbb{R}^d \) as \( a \to \infty \) if and only if \( \mu \) has weak mean \( m_\mu \) for some \( m_\mu^\rho \).

**Proof.** Notice that \( b(\beta) \) is bounded on \((1, 2)\) and that \( \int_{|x|>1} |x| \nu_\rho(dx) < \infty \) by Lemma 3.3. We have

\[
\int_1^a s^{-1} ds \int_{|x|>s} x \nu_\rho(dx) = \int_1^a s^{-1} ds \int_{(1,2)} \Gamma_\rho(d\beta) \int_{\mathbb{R}^d} \xi \lambda_\beta^\mu(d\xi) \int_s^\infty r^{-\beta} dr
\]

\[
= \int_{(1,2)} b(\beta) \Gamma_\rho(d\beta) \int_{\mathbb{R}^d} \xi \lambda_\beta^\mu(d\xi) \int_1^a s^{-\beta} ds = I_1 \quad \text{(say)}
\]
and
\[ \int_{1 < |x| < a} x \nu(dx) = \int_{(1, 2)} \Gamma(d\beta) \int_{S} x \lambda_{\beta}(d\xi) \int_{1}^{a} r^{- \beta} dr = I_{2} \quad \text{say}. \]

Hence
\[ I_{1} - I_{2} = \int_{(1, 2)} (b(\beta) - 1) \Gamma(d\beta) \int_{S} x \lambda_{\beta}(d\xi) \int_{1}^{a} r^{- \beta} dr. \]

Since
\[ \left| (b(\beta) - 1) \int_{1}^{a} r^{- \beta} dr \right| \leq (\beta - 1)^{-1} |b(\beta) - 1| \]
and \( \int_{1}^{a} r^{- \beta} dr \) tends to \((\beta - 1)^{-1}\), \( I_{1} - I_{2} \) is convergent in \( \mathbb{R}^{d} \) as \( a \to \infty \). Hence \( I_{1} \) is convergent if and only if \( I_{2} \) is convergent.

**Lemma 3.5.** Let \( f \) and \( h \) be locally square-integrable functions on \( \mathbb{R}_{+} \). Assume that there is \( s_{0} \in (0, \infty) \) such that \( h(s) = 0 \) for \( s \geq s_{0} \) and that \( \Phi_{h} \) is one-to-one. Then \( \Phi_{f} \Phi_{h} = \Phi_{h} \Phi_{f} \).

**Proof.** Let \( f_{t}(s) = f(s) 1_{[0,t]}(s) \). Then \( \Phi_{f_{t}} \Phi_{h} = \Phi_{h} \Phi_{f_{t}} \) by Lemma 3.6 of Maejima and Sato (2009). Let \( \rho \in \mathcal{D}(\Phi_{f}) \). Then \( \Phi_{f_{t}} \rho \to \Phi_{f} \rho \) as \( t \to \infty \) by the definition of \( \Phi_{f} \). Hence \( \Phi_{h} \Phi_{f_{t}} \rho \to \Phi_{h} \Phi_{f} \rho \) by (3.1) of Maejima and Sato (2009). It follows that \( \Phi_{f_{t}} \Phi_{h} \rho \to \Phi_{h} \Phi_{f} \rho \). Since the convergence of \( \int_{0}^{t} f(s)dX_{s}(\Phi_{h} \rho) \) in law implies its convergence in probability, \( \Phi_{h} \rho \) is in \( \mathcal{D}(\Phi_{f}) \) and \( \Phi_{f} \Phi_{h} \rho = \Phi_{h} \Phi_{f} \rho \). Conversely, suppose that \( \rho \in ID \) satisfies \( \Phi_{h} \rho \in \mathcal{D}(\Phi_{f}) \). Then \( \Phi_{h} \Phi_{f_{t}} \rho \to \Phi_{h} \Phi_{f} \rho \) as \( t \to \infty \). Looking at (3.8) of Maejima and Sato (2009), we see that \( \int_{0}^{\infty} h(s) \neq 0 \) from the one-to-one property of \( \Phi_{h} \). Hence \( \{\Phi_{f_{t}} \rho; t > 0\} \) is precompact by the argument in pp. 138–139 of Maejima and Sato (2009). Hence, again from the one-to-one property of \( \Phi_{h} \), \( \Phi_{f_{t}} \rho \) is convergent as \( t \to \infty \), that is, \( \rho \in \mathcal{D}(\Phi_{f}) \).

**Lemma 3.6.** Let \( f \) be locally square-integrable on \( \mathbb{R}_{+} \). Suppose that there is \( \beta \geq 0 \) such that any \( \mu \in \mathcal{R}(\Phi_{f}) \) has Lévy measure \( \nu_{\mu} \) with a radial decomposition \( (\lambda_{\mu}(d\xi), w_{\xi}(u)du) \) where \( l_{\xi}(u) \) is measurable in \((\xi, u)\) and decreasing on \( \mathbb{R}_{+}^{u} \) in \( u \). Then
\[ \mathcal{R}_{\infty}(\Phi_{f}) \subset \mathcal{R}_{\infty}(\Lambda_{1, -\beta - 1}) = L_{\infty}. \]

**Proof.** Clearly \( l_{\xi}(u) \geq 0 \) for \( \lambda_{\mu} \)-a.e. \( \xi \). Since \( \int_{|x| > 1} \nu_{\mu}(dx) < \infty \), we have \( \lim_{u \to \infty} l_{\xi}(u) = 0 \) for \( \lambda_{\mu} \)-a.e. \( \xi \). Hence we can modify \( l_{\xi}(u) \) in such a way that \( l_{\xi}(u) \) is monotone of order 1 in \( u \in \mathbb{R}_{+}^{u} \). Recall that a function is monotone of order 1 on \( \mathbb{R}_{+}^{u} \) if and only if it is decreasing, right-continuous, and vanishing at infinity (Proposition 2.11 of [S]). Then it follows from Theorem 4.18 or 6.12 of [S] that
\[ \mathcal{R}(\Phi_{f}) \subset \mathcal{R}(\Lambda_{1, -\beta - 1}). \]
Let us write $\Lambda = \Lambda_{1, -\beta - 1}$ for simplicity. We have $\Phi_f \Lambda = \Lambda \Phi_f$ by virtue of Lemma 3.5, since $\Lambda$ is one-to-one (Theorem 6.14 of [S]). If $\Phi_f \Lambda^n = \Lambda^n \Phi_f$ for some integer $n \geq 1$, then

$$\Phi_f \Lambda^{n+1} = \Phi_f \Lambda \Lambda^n = \Lambda \Phi_f \Lambda^n = \Lambda \Lambda^n \Phi_f = \Lambda^{n+1} \Phi_f.$$ 

Hence $\Phi_f \Lambda^n = \Lambda^n \Phi_f$ for $n = 1, 2, \ldots$. Now we claim that

$$\mathcal{R}(\Phi_f^n) \subset \mathcal{R}(\Lambda^n)$$

for $n = 1, 2, \ldots$. Indeed, this is true for $n = 1$ by (3.1); if (3.2) is true for $n$, then any $\mu \in \mathcal{R}(\Phi_f^{n+1})$ has expression

$$\mu = \Phi_f^{n+1} \rho = \Phi_f \Phi_f^n \rho = \Phi_f \Lambda^n \rho' = \Lambda^n \Phi_f \rho' = \Lambda^n \Lambda \rho'' = \Lambda^{n+1} \rho''$$

for some $\rho \in \mathcal{D}(\Phi_f^{n+1})$, $\rho' \in \mathcal{D}(\Lambda^n)$ with $\Phi_f^n \rho = \Lambda^n \rho'$, and $\rho'' \in \mathcal{D}(\Lambda)$ with $\Phi_f \rho' = \Lambda \rho''$, which means (3.2) for $n+1$. It follows from (3.2) that $\mathcal{R}_\infty(\Phi_f) \subset \mathcal{R}_\infty(\Lambda)$. The equality $\mathcal{R}_\infty(\Lambda) = L_\infty$ is from Proposition 2.3.

**Proof of the part related to $\mathcal{R}_\infty(\Psi_\alpha)$ in Theorem 1.1.** The result for $-1 \leq \alpha \leq 0$ is already known (see Propositions 2.1 and 2.2). But the proof below also includes this case. First, using Lemma 3.2, notice that Lemma 3.6 is applicable to $\Phi_f = \Psi_\alpha$ and $\beta = (-\alpha - 1) \vee 0$.

**Case 1 ($-\infty < \alpha < 0$).** We have $\mathcal{D}(\Psi_\alpha) = ID$ in Lemma 3.1. Let us show that

$$\Psi_\alpha(L_\infty) = L_\infty.$$ (3.3)

Let $\rho \in L_\infty$ and $\mu = \Psi_\alpha \rho$. Then for $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}^d$,

$$\nu_\mu(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f_\alpha(s)x) \nu_\rho(dx) = \int_0^\infty t^{-\alpha-1} e^{-t} dt \int_{\mathbb{R}^d} 1_B(tx) \nu_\rho(dx)$$

$$= \int_0^\infty t^{-\alpha-1} e^{-t} dt \int_{(0,2)} \Gamma_\rho(d\beta) \int_S \lambda_\beta^\rho(d\xi) \int_0^\infty 1_B(tx) r^{-\beta-1} dr$$

$$= \int_{(0,2)} \Gamma(\beta - \alpha) \Gamma_\rho(d\beta) \int_S \lambda_\beta^\rho(d\xi) \int_0^\infty 1_B(u\xi) u^{-\beta-1} du.$$ 

Hence $\mu \in L_\infty$ with

$$\Gamma_\mu(d\beta) = \Gamma(\beta - \alpha) \Gamma_\rho(d\beta) \quad \text{and} \quad \lambda_\beta^\mu = \lambda_\beta^\rho.$$ (3.4)

Let us show the converse. Let $\mu \in L_\infty$. In order to find $\rho \in L_\infty$ satisfying $\Psi_\alpha \rho = \mu$, it suffices to choose $\Gamma_\rho$, $\lambda_\beta^\rho$, $A_\rho$, and $\gamma_\rho$ such that (3.4) holds and

$$A_\mu = \int_0^\infty f_\alpha(s)^2 ds A_\rho.$$ (3.5)
It follows from (3.8) that
\[ \gamma_\mu = \int_0^\infty f_\alpha(s)ds \left( \gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|f_\alpha(x)x|\leq 1\}} - 1_{\{|x|\leq 1\}})\nu_\rho(dx) \right) \]  
(see Proposition 3.18 of [S]). This choice is possible, because \( \inf_{\beta \in (0,2)} \Gamma(\beta - \alpha) > 0 \), \( \int_0^\infty f_\alpha(s)ds = \int_0^\infty t^{-\alpha}e^{-t}dt = \Gamma(1 - \alpha) \), \( \int_0^\infty f_\alpha(s)^2ds = \int_0^\infty t^{1-\alpha}e^{-t}dt = \Gamma(2 - \alpha) \), and
\[ \int_0^\infty f_\alpha(s)ds \int_{\mathbb{R}^d} |x| |1_{\{|f_\alpha(x)x|\leq 1\}} - 1_{\{|x|\leq 1\}}|\nu_\rho(dx) \]
\[ = \int_0^\infty t^{-\alpha}e^{-t}dt \int_{\mathbb{R}^d} |x| |1_{\{|x|\leq 1\}} - 1_{\{|x|\leq 1\}}|\nu_\rho(dx) \]
\[ = \int_1^\infty t^{-\alpha}e^{-t}dt \int_{1<|x|\leq 1/t} |x|\nu_\rho(dx) + \int_1^\infty t^{-\alpha}e^{-t}dt \int_{1/t<|x|\leq 1} |x|\nu_\rho(dx) \]
\[ = \int_{|x|>1} |x|\nu_\rho(dx) \int_{0}^{1/|x|} t^{-\alpha}e^{-t}dt + \int_{|x|\leq 1} |x|\nu_\rho(dx) \int_{1/|x|}^\infty t^{-\alpha}e^{-t}dt < \infty, \]
since \( \int_{0}^{1/|x|} t^{-\alpha}e^{-t}dt \sim (1 - \alpha)^{-1}\Gamma(\alpha - 1) \) as \( |x| \to \infty \) and \( \int_{1/|x|}^\infty t^{-\alpha}e^{-t}dt \sim |x|^{-\alpha}e^{-1/|x|} \) as \( |x| \to 0 \). Therefore (3.3) is true. It follows that \( \Psi_\alpha^n(L_\infty) = L_\infty \) for \( n = 1, 2, \ldots \). Hence \( \mathcal{R}_\infty(\Psi_\alpha) \supset L_\infty \). On the other hand, \( \mathcal{R}_\infty(\Psi_\alpha) \subset L_\infty \) by virtue of Lemma 3.6.

Case 2 \((0 \leq \alpha < 1)\). Since \( \mathcal{D}(\Psi_\alpha) \) is as in Lemma 3.1, it follows from Lemma 3.3 that
\[ L_\infty \cap \mathcal{D}(\Psi_\alpha) = \begin{cases} 
\{ \rho \in L_\infty : \int_{(0,2)} \beta^{-2}\Gamma_\rho(d\beta) < \infty \}, & \alpha = 0, \\
\{ \rho \in L_\infty^{(\alpha)} : \int_{(\alpha,2)} (\beta - \alpha)^{-1}\Gamma_\rho(d\beta) < \infty \}, & 0 < \alpha < 1.
\end{cases} \]
We have
\[ \Psi_\alpha(L_\infty \cap \mathcal{D}(\Psi_\alpha)) = L_\infty^{(\alpha,2)}, \]  
(3.7)
where \( L_\infty^{(\alpha,2)} = L_\infty \). Indeed, if \( \rho \in L_\infty \cap \mathcal{D}(\Psi_\alpha) \) and \( \mu = \Psi_\alpha\rho \), then we have \( \mu \in L_\infty^{(\alpha,2)} \) and (3.4), using \( \Gamma(\beta - \alpha) = (\beta - \alpha)^{-1}\Gamma(\beta + 1) \) for \( 0 \leq \alpha < 1 \). Conversely, if \( \mu \in L_\infty^{(\alpha,2)} \), then we can find \( \rho \in L_\infty \cap \mathcal{D}(\Psi_\alpha) \) satisfying \( \mu = \Psi_\alpha\rho \) in the same way as in Case 1; when \( \alpha = 0 \), we have \( \int_{(0,2)} \beta^{-2}\Gamma_\rho(d\beta) < \infty \) since \( \Gamma_\rho(d\beta) = \beta(\Gamma(\beta + 1))^{-1}\Gamma_\mu(d\beta) \) and \( \int_{(0,2)} \beta^{-1}\Gamma_\mu(d\beta) < \infty \). Hence (3.7) holds. Now we have
\[ \Psi_\alpha^n(L_\infty \cap \mathcal{D}(\Psi_\alpha^n)) = L_\infty^{(\alpha,2)} \]  
(3.8)
for \( n = 1, 2, \ldots \). Indeed, it is true for \( n = 1 \) by (3.7) and, if (3.8) is true for \( n \), then
\[ L_\infty^{(\alpha,2)} = \Psi_\alpha^n(L_\infty \cap \mathcal{D}(\Psi_\alpha^n)) = \Psi_\alpha^n(L_\infty^{(\alpha,2)} \cap \mathcal{D}(\Psi_\alpha^n)) \]
\[ = \Psi_\alpha^n(\Psi_\alpha(L_\infty \cap \mathcal{D}(\Psi_\alpha)) \cap \mathcal{D}(\Psi_\alpha^n)) \]
\[ = \Psi_\alpha^n(\Psi_\alpha(L_\infty \cap \mathcal{D}(\Psi_\alpha^{n+1}))) = \Psi_\alpha^{n+1}(L_\infty \cap \mathcal{D}(\Psi_\alpha^{n+1})). \]
It follows from (3.8) that \( L_\infty^{(\alpha,2)} \subset \mathcal{R}_\infty(\Psi_\alpha) \). Next we claim that
\[ \mathcal{R}(\Psi_\alpha) \cap L_\infty \subset L_\infty^{(\alpha,2)}. \]  
(3.9)
Let \( \mu \in \mathcal{R}(\Psi_a) \cap L_\infty \). Then \( \mu \) has a radial decomposition \((\lambda_\mu(d\xi), r^{-\alpha-1} k_\xi^\mu(r)dr)\) with the property stated in Lemma 3.2. On the other hand, from (3.10) by the same argument as in Case 2. Hence

\[
\nu_\mu(B) = \int_{(0,2)} \Gamma_\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1}dr
\]

for \( B \in \mathcal{B}(\mathbb{R}^d) \), as there are a probability measure \( \bar{\lambda}_\mu \) on \( S \) and a measurable family \( \{\Gamma_\xi^\mu\} \) of measures on \((0,2)\) satisfying \( \int_{(0,2)} (\beta^{-1} + (2 - \beta)^{-1}) \Gamma_\xi^\mu(d\beta) = \text{const} \) such that \( \Gamma_\mu(d\beta) \lambda_\beta^\mu(d\xi) = \bar{\lambda}_\mu(d\xi) \Gamma_\xi^\mu(d\beta) \). Hence, by the uniqueness in Proposition 3.1 of [S], there is a positive, finite, measurable function \( B \) for \( \mu \) and \( \lim_{\beta \to \infty} \Gamma_\beta = 0 \) (see Lemma 4.3 of [S]). Since \( \Gamma_\beta((0,\alpha]) = 0 \) for \( \lambda_\mu\)-a.e. \( \xi \). Hence \( \Gamma_\beta((0,\alpha]) = 0 \), that is, \( \mu \in L^{(\alpha,2)} \), proving (3.9). Now, using Lemma 3.6, we obtain \( \mathcal{R}_\infty(\Psi_a) \subset \mathcal{R}(\Psi_a) \cap L_\infty \subset L^{(\alpha,2)} \).

Case 3 (\( \alpha = 1 \)). Let us show that

\[
\Psi_1(L_\infty \cap \mathcal{D}(\Psi_1)) = L^{(1,2)} \cap \{ \mu \in ID: \text{ weak mean 0} \}. \tag{3.10}
\]

Let \( \rho \in L_\infty \cap \mathcal{D}(\Psi_1) \), that is, \( \rho \in L^{(1,2)} \), \( \int_{(1,2)} (\beta - 1)^{-1} \Gamma_\rho (d\beta) < \infty \), \( \int_{\mathbb{R}^d} x\rho(dx) = 0 \), and \( \lim_{a \to \infty} \int_1^a s^{-1}ds \int_{|x| > s} x\nu_\rho(dx) \) exists in \( \mathbb{R}^d \). Let \( \mu = \Psi_1 \rho \). Then, as in Case 1, \( \mu \in L^{(1,2)} \) and (3.4) holds with \( \alpha = 1 \). By Lemma 3.2, \( \mu \) has weak mean 0. Conversely, let \( \mu \in L^{(1,2)} \cap \{ \mu \in ID: \text{ weak mean 0} \} \). Choose \( \rho \in L^{(1,2)}_\infty \) such that \( \Gamma_\rho(d\beta) = (\Gamma(\beta - 1))^{-1} \Gamma_\mu(d\beta) \), \( \lambda_\rho = \lambda_\mu^\rho \), \( A_\rho = A_\mu \), and \( \gamma_\rho = - \int_{|x| > 1} s\nu_\rho(dx) \) (note that \( \int_{(1,2)} (\beta - 1)^{-1} \Gamma_\rho (d\beta) < \infty \) and hence \( \int_{|x| > 1} |x|\nu_\rho(dx) < \infty \) by Lemma 3.3). Then \( \int_{\mathbb{R}^d} x\rho(dx) = 0 \) (see Lemma 4.3 of [S]). Since \( \mu \) has weak mean, \( \int_1^a s^{-1}ds \int_{|x| > s} x\nu_\rho(dx) \) is convergent as \( a \to \infty \) by application of Lemma 3.4 with \( b(\beta) = 1/\Gamma(\beta) \). Hence \( \rho \in \mathcal{D}(\Psi_1) \). We have \( \nu_{\Psi_1 \rho} = \nu_\mu \), \( A_{\Psi_1 \rho} = A_\mu \), and \( \Psi_1 \rho \) has weak mean 0. Among distributions \( \mu' \in ID \) having \( \nu_{\mu'} = \nu_\mu \) and \( A_{\mu'} = A_\mu \), only one distribution has weak mean 0. Hence \( \Psi_1 \rho = \mu \). This proves (3.10). We have

\[
\Psi_1^n(L_\infty \cap \mathcal{D}(\Psi_1^n)) = L^{(1,2)}_\infty \cap \{ \mu \in ID: \text{ weak mean 0} \}, \quad n = 1, 2, \ldots \tag{3.11}
\]

from (3.10) by the same argument as in Case 2. Hence

\[
L^{(1,2)}_\infty \cap \{ \mu \in ID: \text{ weak mean 0} \} \subset \mathcal{R}_\infty(\Psi_1). \tag{3.12}
\]

Next

\[
\mathcal{R}(\Psi_1) \cap L_\infty \subset L^{(1,2)}_\infty \cap \{ \mu \in ID: \text{ weak mean 0} \}. \tag{3.13}
\]
Indeed, $\Re(\Psi_1) \cap L_\infty \subset L_\infty^{(1,2)}$ by the same argument as in Case 2. Any $\mu \in \Re(\Psi_1)$ has weak mean 0 by Lemma 3.2. Now it follows from Lemma 3.6 that

$$\Re_\infty(\Psi_1) \subset L_\infty^{(1,2)} \cap \{\mu \in ID: \text{ weak mean 0}\}. \tag{3.14}$$

*Case 4* ($1 < \alpha < 2$). We show that

$$\Psi_\alpha(L_\infty \cap \mathcal{D}(\Psi_\alpha)) = L_\infty^{(\alpha,2)} \cap \{\mu \in ID: \text{ mean 0}\}. \tag{3.15}$$

Let $\rho \in L_\infty \cap \mathcal{D}(\Psi_\alpha)$, that is, $\rho \in L_\infty^{(\alpha,2)}$, $\int_{(\alpha,2)} (\beta-\alpha)^{-1}\Gamma_\rho(d\beta) < \infty$, and $\int_{R^d} x\rho(dx) = 0$ (Lemmas 3.1 and 3.3). Let $\mu = \Psi_\alpha \rho$. Then $\mu \in L_\infty^{(\alpha,2)}$ and (3.4) holds. Hence $\int_{R^d} |x|\mu(dx) < \infty$ by Lemma 3.3 and $\mu$ has mean 0 by Lemma 3.2. Conversely, if $\mu \in L_\infty^{(\alpha,2)} \cap \{\mu \in ID: \text{ mean 0}\}$, then we can find $\rho \in L_\infty \cap \mathcal{D}(\Psi_\alpha)$ satisfying $\Psi_\alpha \rho = \mu$, similarly to Case 3. Hence (3.15) is true. It follows that

$$\Psi_\alpha^n(L_\infty \cap \mathcal{D}(\Psi_\alpha^n)) = L_\infty^{(\alpha,2)} \cap \{\mu \in ID: \text{ mean 0}\}, \quad n = 1, 2, \ldots$$

similarly to Cases 2 and 3. Hence

$$L_\infty^{(\alpha,2)} \cap \{\mu \in ID: \text{ mean 0}\} \subset \Re_\infty(\Psi_\alpha). \tag{3.16}$$

We can also prove

$$\Re(\Psi_\alpha) \cap L_\infty \subset L_\infty^{(\alpha,2)} \cap \{\mu \in ID: \text{ mean 0}\}$$

similarly to Cases 2 and 3. Hence the reverse inclusion of (3.16) follows from Lemma 3.6.

□

*Proof of the part related to $\Re_\infty(\tilde{\Phi}_{p,\alpha})$ in Theorem 1.1.* We assume $p \geq 1$. Since monotonicity of order $p \in [1,\infty)$ implies monotonicity of order 1 (Corollary 2.6 of [S]), it follows from Lemma 3.2 that Lemma 3.6 is applicable with $\beta = (-\alpha - 1) \lor 0$. Hence $\Re_\infty(\tilde{\Phi}_{p,\alpha}) \subset L_\infty$. If $\rho \in L_\infty \cap \mathcal{D}(\tilde{\Phi}_{p,\alpha})$ and $\tilde{\Phi}_{p,\alpha}\rho = \mu$, then $\rho \in L_\infty^{(\alpha,2)}$ (understand that $L_\infty^{(\alpha,2)} = L_\infty$ for $\alpha \leq 0$) and, noting that

$$\nu_\mu(B) = \int_0^\infty ds \int_{R^d} 1_B(\tilde{\Phi}_{\rho,\alpha}(s)x)\nu_\rho(dx) = \frac{1}{\Gamma(p)} \int_0^1 t^{-\alpha-1}(1-t)^{p-1}dt \int_{R^d} 1_B(tx)\nu_\rho(dx)$$

$$= \frac{1}{\Gamma(p)} \int_0^1 t^{-\alpha-1}(1-t)^{p-1}dt \int_{(\alpha,2)} \Gamma_\rho(d\beta) \int_S \lambda_\beta^p(d\xi) \int_0^\infty 1_B(tr\xi)n^{-\beta-1}dr$$

$$= \int_{(\alpha,2)} \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta - \alpha + p)} \Gamma_\rho(d\beta) \int_S \lambda_\beta^p(d\xi) \int_0^\infty 1_B(u\xi)u^{-\beta-1}du$$

and recalling Lemmas 3.1 and 3.3, we obtain $\mu \in L_\infty^{(\alpha,2)}$ with

$$\Gamma_\rho(d\beta) = \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta - \alpha + p)} \Gamma_\rho(d\beta) \quad \text{and} \quad \lambda_\beta^p = \lambda_\beta^p. \tag{3.17}$$
Now the proof of assertions (i), (ii), and (iv) can be given in parallel to the corresponding assertions for \( \Psi_1 \). Note that, if \(-\infty < \alpha < 1\), then
\[
\int_0^\infty \tilde{f}_{p,\alpha} (s) ds \int_{\mathbb{R}^d} \left| x \right| \left[ 1_{\{ \left| \tilde{f}_{p,\alpha} (s) x \right| \leq 1 \}} - 1 \left| x \right| \leq 1 \right] \nu_\mu (dx) < \infty
\]
similarly. We also use the fact that \( k_\xi^\mu (r) \) vanishes at infinity if it is monotone of order \( p \in [1, \infty) \).

For assertion (iii) in the case \( \alpha = 1 \), we have to find another way, as Lemma 3.4 is not applicable if \( \beta > 1 \). Let us show
\[
\Phi_{p,1} (L_\infty \cap \mathcal{D}(\Phi_{p,1})) = L^{(1,2)}_\infty \cap \{ \mu \in ID: \text{weak mean } 0 \}. 
\] (3.18)
Suppose that \( \rho \in L_\infty \cap \mathcal{D}(\Phi_{p,1}) \) and \( \Phi_{p,1} \rho = \mu \). Then \( \rho \in L^{(1,2)}_\infty \), \( f_{(1,2)} (\beta - 1) \Gamma_\rho (d\beta) < \infty \), \( \mu \in L^{(1,2)}_\infty \) with (3.17), and \( \mu \) has weak mean 0 by Lemma 3.2. Conversely, suppose that \( \mu \in L^{(1,2)}_\infty \) with weak mean 0. As in [S], let \( \mathcal{M}_L \) be the class of infinitely divisible distributions on \( \mathbb{R}^d \) and let \( \Phi_{p,1}^L \) be the transformation of Lévy measures associated with the mapping \( \Phi_{p,1} \). Define \( \Gamma_0 (d\beta) = \frac{\Gamma (\beta - 1 + p)}{\Gamma (\beta - 1)} \Gamma_\mu (d\beta) \). Then \( f_{(1,2)} (2 - \beta)^{-1} \Gamma_0 (d\beta) < \infty \). Define
\[
\nu_0 (B) = \int_{(1,2)} \Gamma_0 (d\beta) \int_S \lambda_\beta (d\xi) \int_0^\infty 1_B (r \xi) r^{-\beta - 1} dr
\]
for \( B \in \mathcal{B}(\mathbb{R}^d) \). We have \( \nu_0 \in \mathcal{M}_L \). We see
\[
\nu_\mu (B) = \int_{(1,2)} \frac{\Gamma (\beta - 1)}{\Gamma (\beta - 1 + p)} \Gamma_0 (d\beta) \int_S \lambda_\beta (d\xi) \int_0^\infty 1_B (u \xi) u^{-\beta - 1} du 
\]
\[
= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B (f_{p,1} (s) x) \nu_0 (dx)
\]
from the calculation above. Since \( \nu_\mu \in \mathcal{M}_L \), we have \( \nu_0 \in \mathcal{D}(\Phi_{p,1}^L) \) and \( \Phi_{p,1}^L \nu_0 = \nu_\mu \). Then it follows from Theorem 4.10 of [S] that \( \nu_\mu \) has a radial decomposition \( (\lambda_\mu (d\xi), u^{-2} k_\xi^\mu (u) du) \) such that \( k_\xi^\mu (u) \) is measurable in \( (\xi, u) \) and, for \( \lambda_\mu \)-a.e. \( \xi \), monotone of order \( p \) in \( u \in \mathbb{R}^d_+ \). Hence \( \mu \in \mathcal{R}(\Phi_{p,1}) \) from Lemma 3.2. Since \( \Phi_{p,1}^L \nu_0 = \nu_\mu \) and \( \Phi_{p,1}^L \) is one-to-one (Theorem 4.9 of [S]), we have \( \mu = \Phi_{p,1} \rho \) for some \( \rho \in \mathcal{D}(\Phi_{p,1}) \) with \( \nu_\rho = \nu_0 \). It follows that \( \rho \in L_\infty \). This finishes the proof of (3.18). Now we can show (3.11)–(3.14) with \( \Phi_{p,1} \) in place of \( \Psi_1 \) similarly to Case 3 in the preceding proof.

Proof of the part related to \( \mathcal{R}_\infty (\Lambda_{q,\alpha}) \) in Theorem 1.1. Since we have Proposition 2.3, it remains only to consider \( \Lambda_{1,1} \). But the assertion for \( \mathcal{R}_\infty (\Lambda_{1,1}) \) is obviously true, since \( \Lambda_{1,1} = \Phi_{1,1} \).

Proof
4. SOME EXAMPLES OF $\mathcal{R}_\infty(\Phi_f)$

We present some examples of $\Phi_f$ for which the class $\mathcal{R}_\infty(\Phi_f)$ is different from those appearing in Theorem 1.1.

Define $T_\alpha$, the dilation by $a \in \mathbb{R} \setminus \{0\}$, as $(T_\alpha \mu)(B) = \int 1_B(ax)\mu(dx) = \mu((1/a)B)$, $B \in \mathcal{B}(\mathbb{R}^d)$, for measures on $\mathbb{R}^d$. Define $P_t$, the raising to the convolution power $t > 0$, in such a way that, for $\mu \in ID$, $P_t \mu$ is an infinitely divisible distribution with characteristic function $\widehat{P_t \mu}(z) = \hat{\mu}(z^t)$. The mappings $T_\alpha$ (restricted to $ID$), $P_t$, and $\Phi_f$ are commutative with each other. A measure $\mu$ on $\mathbb{R}^d$ is called symmetric if $T_{-1} \mu = \mu$. A distribution $\mu$ on $\mathbb{R}^d$ is called shifted symmetric if $\mu = \rho \ast \delta_\gamma$ with some symmetric distribution $\rho$ and some $\delta$-distribution $\delta_\gamma$. Let $ID_{sym} = ID_{sym}(\mathbb{R}^d)$ [resp. $ID_{sh} = ID_{sh}(\mathbb{R}^d)$] denote the class of symmetric [resp. shifted symmetric] infinitely divisible distributions on $\mathbb{R}^d$.

**Example 4.1.** Let $f(s) = b_1[0,a](s) - b_1(a,2a](s)$ with $a > 0$ and $b \neq 0$. Then $\mathcal{R}_\infty(\Phi_f) = ID_{sym}$.

Indeed, for $\rho \in ID$,

$$C_{\Phi_f \rho}(z) = \int_0^a C_{\rho}(bz)ds + \int_a^{2a} C_{\rho}(-bz)ds = aC_{\rho}(bz) + aC_{\rho}(-bz) = C_{\rho, T_{-1} \rho}(z)$$

for $z \in \mathbb{R}^d$, and hence $\Phi_f \rho = P_{a} T_{b} (\rho \ast T_{-1} \rho)$. Define $U \rho = P_{1/2} \rho \ast T_{-1} P_{1/2} \rho$. Then $U \rho \in ID_{sym}$ for any $\rho \in ID$. If $\rho \in ID_{sym}$, then $U \rho = \rho$. Hence $U^n \rho = U \rho$ for $n = 1, 2, \ldots$. Since $\Phi_f = P_{a} T_{b} P_{2} U = P_{2a} T_{b} U$, we have $\Phi_f^n = P_{2a} T_{b} U = U P_{2a} T_{b}$ and $U = \Phi_f^n P_{n/(2a)} T_{1/b}$. Hence $\mathcal{R}_\infty(\Phi_f) = \mathcal{R}(U) = ID_{sym}$.

**Example 4.2.** Let $f(s) = b_1[0,a](s) - b_1(a,a+c](s)$ with $a > 0$, $c > 0$, $a \neq c$, and $b \neq 0$. Then $\mathcal{R}_\infty(\Phi_f) = ID_{sh}$.

To see this, notice that

$$C_{\Phi_f \rho}(z) = aC_{\rho}(bz) + cC_{\rho}(-bz) = (a + c)(a_1 C_{T_{-1} \rho}(z) + (1 - a_1) C_{T_{-1} \rho}(-z))$$

for $\rho \in ID$, where $a_1 = a/(a + c)$. That is, $\Phi_f \rho = P_{a_1} C_{T_{-1}} (P_{a_1} \rho \ast P_{1-a_1} T_{-1} \rho)$. Let us define $V \rho = P_{a_1} \rho \ast P_{1-a_1} T_{-1} \rho$. Note that $V$ is the stochastic integral mapping $\Phi_f$ in the case $a + c = 1$ and $b = 1$. We have

$$V^n \rho = P_{a_n} \rho \ast P_{1-a_n} T_{-1} \rho \tag{4.1}$$

for $n = 1, 2, \ldots$, where $a_n$ is given by $a_n = 1 - a_1 + a_{n-1}(2a_1 - 1)$. Indeed, if (4.1) is true for $n$, then it is true for $n + 1$ in place of $n$, since

$$V^{n+1} \rho = P_{a_n} V \rho \ast P_{1-a_n} T_{-1} V \rho = P_{a_n} V \rho \ast P_{1-a_n} V \rho$$
Hence a from convergent to some \( \nu \) convolution, we obtain \( \delta \). Then \( R \rho \). If \( \rho \in ID_{\text{sym}} \), then \( V \rho = \rho \). Hence \( ID_{\text{sym}} \subset \mathfrak{R}_\infty(V) \). If \( \rho = \delta_\gamma \), then \( V \rho = \delta_{a_1 \gamma} \ast \delta_{-(1-a_1) \gamma} = \delta_{(2a_1-1) \gamma} \). Now \( \delta_\gamma = V \delta_{(1/(2a_1-1)) \gamma} \), since \( a_1 \neq 1/2 \). Hence all \( \delta \)-distributions are in \( \mathfrak{R}(V^n) \) and hence in \( \mathfrak{R}_\infty(V) \). Since \( \mathfrak{R}_\infty(V) \) is closed under convolution, we obtain \( ID_{\text{shift}} \subset \mathfrak{R}_\infty(V) \). To show the converse, assume that \( \mu \in \mathfrak{R}_\infty(V) \). Then \( \mu = V^n \rho_n \) for some \( \rho_n \in ID \). It follows from (4.1) that \( \nu_\mu = a_n \nu_{\rho_n} + (1-a_n) T_1 \nu_{\rho_n} \). Let \( \sigma_n \in ID \) be such that \( (A_{\sigma_n}, \nu_{\sigma_n}, \gamma_{\sigma_n}) = (0, \nu_{\rho_n}, 0) \). It follows from \( a_n = 1 - a_1 + a_{n-1}(2a_1 - 1) \) and from \( 0 < a_n < 1 \) that \( a_n \rightarrow 1/2 \) as \( n \rightarrow \infty \). Hence \( \sigma_n > 1/3 \) for all large \( n \). We see that the set \( \{ \sigma_n : n = 1, 2, \ldots \} \) is precompact, since \( \nu_{\sigma_n} \leq \nu_\mu \leq 3 \nu_\mu \) for all large \( n \). Thus we can choose a subsequence \( \{ \sigma_{n_k} \} \) convergent to some \( \sigma' \in ID \). Since \( \int \varphi(x) \nu_{\sigma_{n_k}}(dx) \rightarrow \int \varphi(x) \nu_{\sigma'}(dx) \) for any bounded continuous function \( \varphi \) which vanishes on a neighborhood of the origin and since \( a_n \rightarrow 1/2 \), we obtain \( \nu_\mu = (1/2) \nu_{\sigma'} + (1/2) T_1 \nu_{\sigma'} \). Hence \( \nu_\mu \) is symmetric. Hence \( \mu \ast \delta \gamma_n \) is symmetric. It follows that \( \mu \in ID_{\text{shift}} \). This proves (4.2) and therefore \( \mathfrak{R}_\infty(\Phi_f) = ID_{\text{shift}} \).

**Example 4.3.** Let \( \alpha < 0 \). Let \( h(s) \) be one of \( f_{\alpha}(s), \tilde{f}_{\rho,\alpha}(s) \), and \( l_{q,\alpha}(s) \) \( p \geq 1, q > 0 \). Let \( s_0 = \sup \{ s : h(s) > 0 \} \). Then \( 0 < s_0 < \infty \). Define

\[
  f(s) = \begin{cases} 
    h(s), & 0 \leq s \leq s_0, \\
    -h(2s_0 - s), & s_0 < s \leq 2s_0, \\
    0, & s > 2s_0.
  \end{cases}
\]

Then \( \mathfrak{R}_\infty(\Phi_f) = L_\infty \cap ID_{\text{sym}} \).

Proof is as follows. First, recall that \( \mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_h) = ID \). We have, for \( \rho \in ID \),

\[
  C_{\Phi_f,\rho}(z) = \int_0^{s_0} C_\rho(h(s)z)ds + \int_{s_0}^{2s_0} C_\rho(-h(2s_0 - s)z)ds \\
  = \int_0^{s_0} C_\rho(h(s)z)ds + \int_{s_0}^{s_0} C_\rho(-h(s)z)ds \\
  = C_{\Phi_h,\rho}(z) + C_{\Phi_h, T_1 \rho}(z).
\]

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It follows that $\Phi_f \rho = \Phi_h(\rho \ast T_{-1}\rho) = \Phi_hP_2U\rho = UP_2\Phi_h\rho$, where $U$ is the mapping used in Example 4.1. It follows that $\Phi_f^n = \Phi_h^n P_2^n U = UP_2^n \Phi_h^n$ for $n = 1, 2, \ldots$. Hence $\mathcal{R}(\Phi_f^n) \subset \mathcal{R}(\Phi_h^n) \cap ID_{sym}$. Conversely, assume that $\rho \in \mathcal{R}(\Phi_h^n) \cap ID_{sym}$. Then $\mu = \Phi_h^n \rho$ for some $\rho$ and $T_{-1}\mu = \Phi_h^n T_{-1}\rho$. Since $\Phi_h$ is one-to-one (see [S]), we have $\rho = T_{-1}\rho$. Hence $\Phi_f^n \rho = \Phi_h^n P_2^n U \rho = \Phi_h^n P_2^n \rho = P_2^n \mu$ and thus $\mu = \Phi_f^n P_1^n \rho \in \mathcal{R}(\Phi_f^n)$. In conclusion, $\mathcal{R}(\Phi_f^n) = \mathcal{R}(\Phi_h^n) \cap ID_{sym}$ and hence $\mathcal{R}_\infty(\Phi_f) = \mathcal{R}_\infty(\Phi_h) \cap ID_{sym} = L_\infty \cap ID_{sym}$.

**Example 4.4.** Let $h(s)$ and $s_0$ be as in Example 4.3. Define

$$f(s) = \begin{cases} h(s_0 - s), & 0 \leq s \leq s_0, \\ h(s - s_0), & s_0 < s \leq 2s_0, \\ -h(3s_0 - s), & 2s_0 < s \leq 3s_0, \\ 0, & s > 3s_0. \end{cases}$$

Then $\mathcal{R}_\infty(\Phi_f) = L_\infty \cap ID_{shift}$.

To see this, notice that

$$C_{\Phi_f \rho}(z) = \int_0^{s_0} C_\rho(h(s_0 - s)z)ds + \int_{s_0}^{2s_0} C_\rho(h(s_0 - s)z)ds$$

$$+ \int_{2s_0}^{3s_0} C_\rho(-h(3s_0 - s)z)ds$$

$$= \int_0^{s_0} C_\rho(h(s)z)ds + \int_{s_0}^{2s_0} C_\rho(h(s)z)ds + \int_{2s_0}^{3s_0} C_\rho(-h(s)z)ds$$

$$= 2C_{\Phi_h \rho}(z) + C_{\Phi_h \rho}(-z)$$

$$= 3\left(\frac{2}{3}C_{\Phi_h \rho}(z) + \frac{1}{3}C_{\Phi_h \rho}(-z)\right).$$

Hence $\Phi_f \rho = P_3 V \Phi_h \rho$, where $V \rho = P_{2/3} \rho \ast P_{1/3} T_{-1} \rho$. This mapping $V$ is a special case of $V$ in Example 4.2 with $a_1 = 2/3$. Hence (4.1) holds with $a_n = 2^{-1}(1 + 3^{-n})$ and $1 - a_n = 2^{-1}(1 - 3^{-n})$. Now $\Phi_f^n = P_3^n V^n \Phi_h^n = \Phi_h^n P_3^n V^n = V^n P_3^n \Phi_h^n$. Hence $\mathcal{R}(\Phi_f^n) \subset \mathcal{R}(\Phi_h^n) \cap \mathcal{R}(V^n)$. It follows that $\mathcal{R}_\infty(\Phi_f) \subset \mathcal{R}_\infty(\Phi_h) \cap \mathcal{R}_\infty(V) = L_\infty \cap ID_{shift}$ from Theorem 1.1 and (4.2). Let us also show the converse inclusion $L_\infty \cap ID_{shift} \subset \mathcal{R}_\infty(\Phi_f)$. It is enough to show

$$\mathcal{R}(\Phi_h^n) \cap ID_{shift} \subset \mathcal{R}(\Phi_f^n). \quad (4.3)$$

For any $\gamma \in \mathbb{R}^d$ we have

$$C_{\Phi_h,\delta,\gamma}(z) = \int_0^{s_0} C_{\delta,\gamma}(h(s)z)ds = i \int_0^{s_0} \langle \gamma, h(s)z \rangle ds = ic\langle \gamma, z \rangle = C_{\delta,\gamma}(z),$$

where $c = \int_0^{s_0} h(s)ds > 0$. That is, $\Phi_h \delta = \delta_{c\gamma}$. Hence $\Phi_f \delta = P_3 \Phi_h V \delta = P_3 \Phi_h(\delta_{(2/3)\gamma} \ast \delta_{-1/3}) = \Phi_h \delta = \delta_{c\gamma}$. Hence $\Phi_f^n \delta = \delta_{c^n \gamma}$ and $\delta = \Phi_f^n \delta_{c^{-n} \gamma}$. Hence
all \( \delta \)-distributions are in \( \mathcal{R}(\Phi^n_f) \). Similarly all \( \delta \)-distributions are in \( \mathcal{R}(\Phi^n_h) \). Let 
\[ \mu \in \mathcal{R}(\Phi^n_h) \cap ID_{\text{sym}}^{\text{shift}}. \]
Then \( \mu + \delta \gamma \in \mathcal{R}(\Phi^n_h) \cap ID_{\text{sym}} \) for some \( \gamma \). Letting \( \mu' = \mu + \delta \gamma \), we have \( \mu' = \Phi^n_h \rho' \) for some \( \rho' \). Since \( \mu' = T^{-1} \mu' = \Phi^n_h T^{-1} \rho' \), we have \( \rho' = T^{-1} \rho' \) from the one-to-one property of \( \Phi_h \). Thus \( V^n \rho' = \rho' \) and \( \Phi^n_f \rho' = \Phi^n_h P_3^n \rho' = P^n \rho' \). Hence 
\[ \mu' = P^n_1 \Phi^n_f \rho' = \Phi^n_f P^n_1 \rho' \in \mathcal{R}(\Phi^n_f). \]
It follows that \( \mu = \mu' + \delta \gamma \in \mathcal{R}(\Phi^n_f) \). This proves (4.3). Hence \( \mathcal{R}_\infty(\Phi_f) = L_\infty \cap ID_{\text{sym}}^{\text{shift}}. \)

**Example 4.5.** Let \( b > 1 \). Let 
\[ f(s) = b 1_{[0,1]}(s) + 1_{[1,2]}(s). \]
Then 
\[ L_\infty(b) \subset \mathcal{R}_\infty(\Phi_f) \subset \not\subset ID. \]
We do not know whether \( \mathcal{R}_\infty(\Phi_f) \) equals \( L_\infty(b) \). Here \( L_\infty(b) \) is the \( b \)-semi-analogue of the class \( L_\infty \), mentioned in Section 2.

Let us show that 
\[ L_\infty(b) \subset \mathcal{R}_\infty(\Phi_f). \]
For \( 0 < \alpha \leq 2 \) define 
\( \mathcal{S}_\alpha(b) = \mathcal{S}_\alpha(b, \mathbb{R}^d) \) as follows: \( \rho \in \mathcal{S}_\alpha(b) \) if and only if \( \rho \) is a \( \delta \)-distribution or a non-trivial \( \alpha \)-semi-stable distribution with \( b \) as a span, that is,
\[ \mathcal{S}_\alpha(b) = \{ \rho \in ID: P_{b \rho} = T_{b \rho} + \delta \gamma \text{ for some } \gamma \in \mathbb{R}^d \}. \]
We have 
\[ C_{\Phi_f \rho}(z) = C_\rho(bz) + C_\rho(z) \text{ for } \rho \in ID, \]
that is, \( \Phi_f \rho = T_{b \rho} + \delta \gamma \). If \( \rho \in \mathcal{S}_\alpha(b) \) with 
\( P_{b \rho} \rho = T_{b \rho} + \delta \gamma \), then \( \mu = \Phi_f \rho \) satisfies 
\( \mu = T_{b \rho} + \delta \gamma \) \( \text{ for } \rho = P_{b \rho + \delta \gamma} \). If \( \mu \in \mathcal{S}_\alpha(b) \) with 
\( P_{b \rho} \mu = T_{b \rho} + \delta \gamma \), then \( \mu = \Phi_f \rho \) for 
\( \rho = P_{b \rho} \rho + \delta \gamma \). Thus 
\[ \mathcal{S}_\alpha(b) = \mathcal{S}_\alpha(b). \]
Hence \( \mathcal{S}_\alpha(b) \subset \mathcal{R}(\Phi^n_f) \) for \( 0 < \alpha \leq 2 \) and \( n = 1, 2, \ldots. \) It follows from Proposition 3.2 of Maejima and Sato (2009) that \( \mathcal{R}(\Phi^n_f) \) is closed under convolution and weak convergence. Hence 
\[ L_\infty(b) \subset \mathcal{R}(\Phi^n_f) \]
and thus \( L_\infty(b) \subset \mathcal{R}_\infty(\Phi_f) \). In order to show 
\[ \mathcal{R}_\infty(\Phi_f) \subset \not\subset ID, \]
let \( \mu \) be such that \( \nu_{\mu} = \delta \alpha \) with \( \alpha \neq 0 \). Suppose that \( \mu = \Phi_f \rho \) for some \( \rho \in ID. \)
Then \( \nu_{\rho} = T_{b \rho} + \nu_{\rho} \). If \( \nu_{\rho} \neq 0 \), then the support of \( \nu_{\rho} \) contains at least one point \( \alpha' \neq 0 \) and hence the support of \( \nu_{\mu} \) contains at least two points \( \{ \alpha', b \alpha' \} \), which is absurd. If \( \nu_{\rho} = 0 \), then \( \nu_{\mu} = 0 \), which is also absurd. Therefore \( \mu \notin \mathcal{R}(\Phi_f) \) and hence 
\( \mu \notin \mathcal{R}_\infty(\Phi_f). \)

5. **Concluding remarks**

The limit class \( \mathcal{R}_\infty(\Phi_f) \) is not known in many cases. For instance it is not known for the following choices of \( f(s) \): 
\( l_{q,1}(s) \) with \( q \in (0, 1) \cup (1, \infty) \) in \( [S]; \)
\( \tilde{f}_{p,\alpha}(s) \) with \( p \in (0, 1) \) and \( \alpha \in (-\infty, 2) \) in \( [S]; \)
\( \cos(2^{-1} \pi s) \) in Maejima et al. (2010b); 
\( e^{-s} l_{[0,c]}(s) \) with \( c \in (0, \infty) \) in Pedersen and Sato (2005); 
\( G_{\alpha,\beta}^n(s) \) with \( \alpha \in [1, 2] \) and \( \beta > 0 \) satisfying \( \alpha = 1 + n \beta \) for some \( n = 0, 1, \ldots \) in Maejima and Ueda (2010b). Another instance is \( \Phi_f = T^n \) with \( \alpha \in (0, 1) \) related to the Mittag-Leffler function, introduced in Barndorff-Nielsen and Thorbjørnsen (2006).
Consider, as in Sato (2007), a stochastic integral mapping

\[ \Phi_f \rho = \mathcal{L} \left( \int_{0+}^{a} f(s)dX_s^{(\rho)} \right) \]

with \(0 < a < \infty\) for a function \(f(s)\) locally square-integrable on the interval \((0, a]\) and study \(\mathcal{R}_\infty(\Phi_f) = \bigcap_{n=1}^\infty \mathcal{R}(\Phi_n^\rho)\). Under appropriate choices of \(f\) we obtain \(\mathcal{R}_\infty(\Phi_f)\) equal to \(L^{(0,\alpha)}_\infty \cap ID_0\) with \(\alpha \in (1, 2)\), \(L^{(0,\alpha)}_\infty \cap ID_0 \cap \{\mu \in ID : \mu \text{ has drift 0}\}\) with \(\alpha \in (0, 1)\), or a certain subclass of \(L^{(0,1)}_\infty \cap ID_0\). This will be shown in a forthcoming paper.

It is an interesting problem what other classes can appear as \(\mathcal{R}_\infty(\Phi_f)\).

References


M. Maejima and Y. Ueda. The relation between $\lim_{m \to \infty} R(\Psi_m^{\alpha+1})$ and $\lim_{m \to \infty} R(\Phi_m^{\alpha+1})$ (2009b). Private Communication.


