LEBESGUE DECOMPOSITION BETWEEN TWO PATH SPACE MEASURES INDUCED BY LÉVY PROCESSES

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This is a report of some results in the lecture notes [9].

1. Hellinger–Kakutani inner product and distance

As Kakutani [3] (1948), Brody [1] (1971), and Newman [5], [6] (1972, 73) have shown, the Hellinger–Kakutani inner product and distance are powerful tool in the problems of absolute continuity and singularity.

Given a measure \( \mu \) and a nonnegative measurable function \( f \), we denote by \( f\mu \) the measure defined as

\[
(f\mu)(B) = \int_B f d\mu.
\]

Let \( \rho_1, \rho_2 \) be \( \sigma \)-finite measures on a measurable space \((\Theta, \mathcal{B})\). The following notation is used: \( \rho_2 \ll \rho_1 \) means that \( \rho_2 \) is absolutely continuous with respect to \( \rho_1 \); \( \rho_2 \perp \rho_1 \) means that \( \rho_2 \) and \( \rho_1 \) are mutually singular; \( \rho_2 \approx \rho_1 \) means that \( \rho_2 \ll \rho_1 \) and \( \rho_1 \ll \rho_2 \).

**Definition 1.1.** Let \( 0 < \alpha < 1 \). The Hellinger–Kakutani inner product of \( \rho_1 \) and \( \rho_2 \) of order \( \alpha \) is the measure \( H_\alpha(\rho_1, \rho_2) \) defined by

\[
H_\alpha(\rho_1, \rho_2) = \left( \frac{d\rho_1}{d\rho} \right)^\alpha \left( \frac{d\rho_2}{d\rho} \right)^{1-\alpha} \rho, \quad 0 < \alpha < 1,
\]

where \( \rho_1 \ll \rho \) and \( \rho_2 \ll \rho \). It is independent of the choice of \( \rho \). Sometimes we write

\[
dH_\alpha(\rho_1, \rho_2) = (d\rho_1)^\alpha (d\rho_2)^{1-\alpha}.
\]

The total mass of \( H_\alpha(\rho_1, \rho_2) \) is written as

\[
h_\alpha(\rho_1, \rho_2) = \int_\Theta dH_\alpha(\rho_1, \rho_2).
\]

**Remark 1.2.** We have

\[
H_\alpha(\rho_1, \rho_2) \leq \alpha \rho_1 + (1-\alpha)\rho_2.
\]

We have \( \rho_1 \perp \rho_2 \) if and only if \( h_\alpha(\rho_1, \rho_2) = 0 \).
Definition 1.3. Write
\begin{align}
C(\rho_1) &= C_\rho(\rho_1) = \left\{ \theta \in \Theta : \frac{d\rho_1}{d\rho} > 0 \right\}, \\
C(\rho_2) &= C_\rho(\rho_2) = \left\{ \theta \in \Theta : \frac{d\rho_2}{d\rho} > 0 \right\},
\end{align}
where $\rho_1 \ll \rho$ and $\rho_2 \ll \rho$. $C(\rho_1)$ is the carrier of $\rho_1$ relative to $\rho$ and $C(\rho_2)$ is the carrier of $\rho_2$ relative to $\rho$.

Remark 1.4. Let $\rho_2 = \rho_2^{ac} + \rho_2^s$ be the Lebesgue decomposition of $\rho_2$ with respect to $\rho_1$, where $\rho_2^{ac}$ is absolutely continuous and $\rho_2^s$ is singular with respect to $\rho_1$. Then,
\begin{align}
\rho_2^{ac} &= 1_{C(\rho_1)}(\rho_2) \quad \text{and} \quad \rho_2^s = 1_{C(\rho_2)}(\rho_2).
\end{align}
If $\rho_1$ and $\rho_2$ are finite, then
\begin{align}
\lim_{\alpha \downarrow 0} h_\alpha(\rho_1, \rho_2) &= \rho_2^{ac}(\Theta), \\
\lim_{\alpha \uparrow 1} h_\alpha(\rho_1, \rho_2) &= \rho_1(\Theta).
\end{align}

Definition 1.5. Let $0 < \alpha < 1$. Define
\begin{align}
K_\alpha(\rho_1, \rho_2) &= \alpha \rho_1 + (1 - \alpha) \rho_2 - H_\alpha(\rho_1, \rho_2),
\end{align}
which is a $\sigma$-finite measure. The total mass
\begin{align}
k_\alpha(\rho_1, \rho_2) &= \int_{\Theta} dK_\alpha(\rho_1, \rho_2)
\end{align}
is called the Hellinger–Kakutani distance of order $\alpha$ between $\rho_1$ and $\rho_2$.

Remark 1.6. Sometimes we write
\begin{align}
dK_{1/2}(\rho_1, \rho_2) &= \frac{1}{2} (\sqrt{d\rho_1} - \sqrt{d\rho_2})^2.
\end{align}
Let $\|\rho_1 - \rho_2\|$ be the total variation norm of $\rho_1 - \rho_2$, admitting infinity. Then
\begin{align}
\|\rho_1 - \rho_2\| &\geq 2 k_{1/2}(\rho_1, \rho_2).
\end{align}
If $\rho_1$ and $\rho_2$ are finite measures, then
\begin{align}
\|\rho_1 - \rho_2\| &\leq c k_{1/2}(\rho_1, \rho_2)^{1/2},
\end{align}
where $c = 2(\rho_1(\Theta) + \rho_2(\Theta))^{1/2}$.

Lemma 1.7. Assume that
\begin{align}
k_\alpha(\rho_1, \rho_2) < \infty
\end{align}
for some $0 < \alpha < 1$. Then (1.15) is true for all $0 < \alpha < 1$ and

\begin{align}
(1.16) \quad & \lim_{\alpha \to 0} k_{\alpha}(\rho_1, \rho_2) = \rho_2(C(\rho_1)^\gamma) < \infty, \\
(1.17) \quad & \lim_{\alpha \to 1} k_{\alpha}(\rho_1, \rho_2) = \rho_1(C(\rho_2)^\gamma) < \infty.
\end{align}

Lemma 1.8. For $j = 1, 2$, let $\nu_j$ be $\sigma$-finite measures on $\mathbb{R}^d$ satisfying $\nu_j(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu_j < \infty$. If $k_{\alpha}(\nu_1, \nu_2) < \infty$, then

\begin{align}
(1.18) \quad & \int_{|x| \leq 1} |x|d\nu_1 - \nu_2 < \infty, \\
(1.19) \quad & \int_{|x| \leq 1} |x|d\nu_j - H_{\alpha}(\nu_1, \nu_2) < \infty, \quad j = 1, 2.
\end{align}

2. General Theory

Let $D = D([0, \infty), \mathbb{R}^d)$, $X_t(\omega) = \omega(t)$ for $\omega \in D$, $\mathcal{F}_t = \sigma(X_s: 0 \leq s \leq t)$, and $\mathcal{F}^0 = \sigma(X_s: 0 \leq s < \infty)$. Any Lévy process on $\mathbb{R}^d$ can be realized as $([X_t], P)$, where $P$ is a probability measure on $(D, \mathcal{F}^0)$. It is said to have the generating triplet $(A, \nu, \gamma)$ if

\begin{align}
E^P[e^{i\langle z, X_t \rangle}] = \exp \left[ i \left( -\frac{1}{2} \langle Az, z \rangle + i \langle \gamma, z \rangle \\
+ \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_{\{|x| \leq 1\}}(x))\nu(dx) \right) \right]
\end{align}

for $z \in \mathbb{R}^d$, where $A$ is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and $\nu$ is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. (Lévy–Khintchine representation)

Let $([X_t], P)$ be a Lévy process with generating triplet $(A, \nu, \gamma)$, where $P$ is a probability measure on $(D, \mathcal{F}^0)$. For any $G \in \mathcal{B}_{(0, \infty) \times (\mathbb{R}^d \setminus \{0\})}$ let $J(G, \omega)$ be the number of $s > 0$ such that $(s, X_s(\omega) - X_{s-}(\omega)) \in G$. Then $J(G)$ has Poisson distribution with mean $\bar{\nu}(G)$, where $\bar{\nu} = ds \times \nu(dx)$. If $G_1, \ldots, G_n$ are disjoint, then $J(G_1), \ldots, J(G_n)$ are independent. We can define

\begin{align}
X'_t(\omega) = \lim_{\epsilon \to 0} \int_{(0,t] \times \{|x| \leq 1\}} \{xJ(d(s, x), \omega) - x\bar{\nu}(d(s, x))\} \\
+ \int_{(0,t] \times \{|x| > 1\}} xJ(d(s, x), \omega),
\end{align}

where the convergence in the right-hand side is uniform in $t$ in any finite time interval, $P$-a.s. Define

\begin{align}
X''_t(\omega) = X_t(\omega) - X'_t(\omega).
\end{align}
Then, \( (\{X'_t\}, P) \) and \( (\{X''_t\}, P) \) are independent Lévy processes with generating triplets \((0, \nu, 0)\) and \((A, 0, \gamma)\), respectively. We call \( (\{X'_t\}, P) \) and \( (\{X''_t\}, P) \) the jump part and the continuous part of \( (\{X_t\}, P) \), respectively. (Lévy–Itô decomposition)

Consider two Lévy processes \( (\{X_t\}, P_1) \) and \( (\{X_t\}, P_2) \) on \( \mathbb{R}^d \), where \( P_1 \) and \( P_2 \) are probability measures on \( (\mathcal{D}, \mathcal{F}_0) \). For \( j = 1, 2 \) denote the generating triplet of \( (\{X_t\}, P_j) \) by \((A_j, \nu_j, \gamma_j)\). When \( A_1 = A_2 \), we write \( A_1 = A_2 = A \). In this case define \( \mathcal{R}(A) = \{ Ax : x \in \mathbb{R}^d \} \). Denote the restriction of \( P_j \) to \( \mathcal{F}_t \) by \( P^t_j \).

The following Theorem A was given by Newman [5], [6] (1972, 73). He essentially proved also Corollaries 2.1–2.5.

**Theorem A.** (i) Suppose that

\[(\text{NS}) \quad k_\alpha(\nu_1, \nu_2) < \infty, \quad A_1 = A_2, \quad \text{and} \quad \gamma_{21} \in \mathcal{R}(A),\]

where

\[
\gamma_{21} = \gamma_2 - \gamma_1 - \int_{|x| \leq 1} x d(\nu_2 - \nu_1). \tag{2.4}
\]

Then

\[
H_\alpha(P^t_1, P^t_2) = e^{-tL_\alpha} P^t_\alpha \quad \text{for} \quad 0 < t < \infty, \quad 0 < \alpha < 1, \tag{2.5}
\]

where

\[
L_\alpha = \frac{1}{2} \alpha (1 - \alpha) \langle \eta, A\eta \rangle + k_\alpha(\nu_1, \nu_2)
\]

with \( \eta \) satisfying \( A\eta = \gamma_{21} \), and \( P_\alpha \) is the probability measure for which \( (\{X_t\}, P_\alpha) \) is the Lévy process generated by \((A, H_\alpha(\nu_1, \nu_2), \gamma_\alpha)\) with

\[
\gamma_\alpha = \alpha \gamma_1 + (1 - \alpha) \gamma_2 - \int_{|x| \leq 1} x dK_\alpha(\nu_1, \nu_2). \tag{2.7}
\]

(ii) Suppose that (NS) is not satisfied, then

\[
H_\alpha(P^t_1, P^t_2) = 0 \quad \text{for} \quad 0 < t < \infty, \quad 0 < \alpha < 1. \tag{2.8}
\]

**Corollary 2.1.** The following three conditions are equivalent.

1. \( P^t_2 \) and \( P^t_1 \) are not mutually singular for some \( 0 < t < \infty \).
2. \( P^t_2 \) and \( P^t_1 \) are not mutually singular for any \( 0 < t < \infty \).
3. Condition (NS) is satisfied.

**Corollary 2.2.** If \( P^t_2 \) and \( P^t_1 \) are not mutually singular, then

\[
\nu_2(C(\nu_1)^c) < \infty \quad \text{and} \quad \nu_1(C(\nu_2)^c) < \infty \tag{2.9}
\]

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and
\[
P_2^t(C(P_1^t)) = e^{-t\nu_2(C(\nu_1)^c)} \quad \text{and} \quad P_1^t(C(P_2^t)) = e^{-t\nu_1(C(\nu_2)^c)}.
\] (2.10)

**Corollary 2.3.** The following three conditions are equivalent.

1. \( P_2^t \ll P_1^t \) for some \( 0 < t < \infty \).
2. \( P_2^t \ll P_1^t \) for any \( 0 < t < \infty \).
3. \( \nu_2 \ll \nu_1 \) and Condition (NS) is satisfied.

**Corollary 2.4.** The following three conditions are equivalent.

1. \( P_2^t \approx P_1^t \) for some \( 0 < t < \infty \).
2. \( P_2^t \approx P_1^t \) for any \( 0 < t < \infty \).
3. \( \nu_2 \approx \nu_1 \) and Condition (NS) is satisfied.

**Corollary 2.5 (dichotomy).** If \( \nu_2 \approx \nu_1 \), then either \( P_2^t \approx P_1^t \) for all \( t > 0 \) or \( P_2^t \perp P_1^t \) for all \( t > 0 \).

The next corollary considers \( P_1 \) and \( P_2 \) on the whole \( \mathcal{F}^0 \).

**Corollary 2.6.** \( P_2 \perp P_1 \) if \( P_2 \neq P_1 \).

Let us prepare Theorem B. In the case where \( P_2^t \approx P_1^t \), (i) and (iii) of Theorem B are trivial and (ii) was shown by Skorokhod [10], [11], [12] (1957, 60, 61) and Kunita and S. Watanabe [4] (1967), a proof of which is given in Sato [8] (1999). But here we do not assume \( P_2^t \approx P_1^t \), nor \( P_2^t \ll P_1^t \).

Let \( P_2^t = (P_2^t)^{ac} + (P_2^t)^s \) be the Lebesgue decomposition of \( P_2^t \) with respect to \( P_1^t \), and \( \nu_2 = \nu_2^{ac} + \nu_2^s \) be the Lebesgue decomposition of \( \nu_2 \) with respect to \( \nu_1 \). Let \( \nu = \nu_1 + \nu_2 \). Choose the versions
\[
\frac{d\nu_j}{d\nu} = f_j \quad \text{for } j = 1, 2
\] (2.11)
satisfying
\[
f_1 \geq 0, \quad f_2 \geq 0, \quad \text{and } f_1 + f_2 = 1 \quad \text{everywhere on } \mathbb{R}^d.
\] (2.12)

Denote
\[
C_1 = \{f_1 = 1 \text{ and } f_2 = 0\},
\]
\[
C_2 = \{f_1 = 0 \text{ and } f_2 = 1\},
\]
(2.13)
\[
C = \{f_1 > 0 \text{ and } f_2 > 0\}.
\]

Thus
\[
\nu_2^{ac} = 1_C \nu_2 \quad \text{and} \quad \nu_2^s = 1_{C_1 \cup C_2} \nu_2 = 1_{C_1 \cup C_2} \nu_2
\] (2.14)
and $d\nu_2^{ac}/d\nu_1$ has the following version:

\begin{equation}
\frac{d\nu_2^{ac}}{d\nu_1} = \begin{cases}
    f_2/f_1 & \text{on } C \\
    0 & \text{on } C_1 \cup C_2.
\end{cases}
\end{equation}

Define

\begin{equation}
g(x) = \begin{cases}
    \log(f_2/f_1) & \text{on } C \\
    -\infty & \text{on } C_1 \cup C_2,
\end{cases}
\end{equation}

\begin{equation}
\tilde{g}(x) = \begin{cases}
    g(x) & \text{on } C \\
    0 & \text{on } C_1 \cup C_2.
\end{cases}
\end{equation}

**Lemma 2.7.** Suppose that $P_t^2$ and $P_t^1$ are not mutually singular for $0 < t < \infty$. Then the following are true.

(i) We can define

\begin{equation}
V_t = \lim_{\varepsilon \downarrow 0} \left( \sum_{(s,X_s-X_{s-}) \in (0,t] \times \{|x|>\varepsilon\}} \tilde{g}(X_s-X_{s-}) - t \int_{|x|>\varepsilon} (e^{g(x)}-1)\nu_1(dx) \right); \tag{2.18}
\end{equation}

the right-hand side exists $P_1$-a.s. and the convergence is uniform on any bounded time interval $P_1$-a.s.

(ii) Let $\eta \in \mathbb{R}^d$ and define

\begin{equation}
U_t^{(n)} = \langle \eta, X_t'' \rangle - \frac{t}{2} \langle \eta, A\eta \rangle - t\langle \gamma_1, \eta \rangle + V_t, \tag{2.19}
\end{equation}

where $\{X_t''\}$ is the continuous part of $\{X_t, P_1\}$. Then $\{U_t^{(n)} : t \geq 0\}$ is, under $P_1$, a Lévy process on $\mathbb{R}$ with generating triplet $(A_t^{(n)}, \nu_t, \gamma_t^{(n)})$ given by

\begin{align}
A_t^{(n)} &= \langle \eta, A\eta \rangle, \\
\nu_t(B) &= \int_{\mathbb{R}^d} 1_B(g(x))\nu_1(dx) \quad \text{for } B \in \mathcal{B}_{\mathbb{R}\setminus\{0\}}, \\
\gamma_t^{(n)} &= -\frac{1}{2} \langle \eta, A\eta \rangle - \int_{\mathbb{R}^d} (e^{g(x)}-1 - g(x)1_{|g(x)| \leq 1}(x))\nu_1(dx). \tag{2.22}
\end{align}

The processes $\{U_t^{(n)} : t \geq 0\}$ and $\{J((0,t] \times (C_1 \cup C_2)) : t \geq 0\}$ are independent under $P_1$.

Define $\Lambda_t \in \mathcal{F}_t^0$ by

\begin{equation}
\Lambda_t = \begin{cases}
    J((0,t] \times (C_1 \cup C_2)) = 0 \quad & \text{if } t > 0, \\
    \{X_s - X_{s-} \notin C_1 \cup C_2 \text{ for all } s \in (0,t]\} \quad & \text{if } t = 0.
\end{cases}
\end{equation}

To the best of our knowledge, the following results are new.
Theorem B. Suppose that $P^t_2$ and $P^t_1$ are not mutually singular for $0 < t < \infty$. Then the following are true.

(i) For $0 < t < \infty$ the Lebesgue decomposition of $P^t_2$ with respect to $P^t_1$ is given by

\begin{align}
(P^t_2)^{ac} &= 1_{\Lambda} P^t_2, \\
(P^t_2)^s &= 1_{D \setminus \Lambda} P^t_2.
\end{align}

We have $P_1(\Lambda_t) = e^{-t\nu_1(C_1)}$ and $P_2(\Lambda_t) = e^{-t\nu_2(C_2)}$.

(ii) The Radon–Nikodým density of $(P^t_2)^{ac}$ is given by

\begin{align}
\frac{d(P^t_2)^{ac}}{dP^t_1} &= e^{-t\nu_2(C_2)} + U_t 1_{\Lambda_t},
\end{align}

where $U_t = U^{(n)}_t$ with $\eta$ satisfying $A\eta = \gamma_{21}$.

(iii) Let $Q$ be the probability measure on $(D, \mathcal{F}_0)$ for which $(\{X_t\}, Q)$ is the Lévy process with generating triplet $(A, \nu_2^{ac}, \gamma_2 - \int_{|x| \leq 1} x d\nu_2^s)$. Then

\begin{align}
(P^t_2)^{ac} &= e^{-t\nu_2(C_2)} Q^t.
\end{align}

Proofs of all results are given in [9].

3. Examples

1. Gaussian case. This is a special case of the results of Cameron and Martin. Suppose that $(\{X_t\}, P_1)$ and $(\{X_t\}, P_2)$ are Lévy processes on $\mathbb{R}^d$ with generating triplets $(A_1, 0, \gamma_1)$ and $(A_2, 0, \gamma_2)$, respectively. Then, for any fixed $t$, Theorems A and B give the following.

(i) The dichotomy holds: either $P^t_2 \approx P^t_1$ or $P^t_2 \perp P^t_1$.

(ii) $P^t_2 \approx P^t_1$ if and only if

\begin{align}
A_1 &= A_2 \quad \text{and} \quad \gamma_2 - \gamma_1 \in \mathcal{R}(A).
\end{align}

(iii) If $P^t_2 \approx P^t_1$, then, for $0 < \alpha < 1$,

\begin{align}
H_\alpha(P^t_1, P^t_2) &= e^{-tL_\alpha} P^t_\alpha,
\end{align}

where $P_\alpha$ is the probability measure for which $(\{X_t\}, P_\alpha)$ is the Lévy process generated by $(A, 0, \gamma_\alpha)$ with $\gamma_\alpha = \alpha \gamma_1 + (1 - \alpha) \gamma_2$, and

\begin{align}
L_\alpha &= \frac{1}{2} \alpha (1 - \alpha) \langle \eta, A\eta \rangle
\end{align}

with $\eta$ satisfying $A\eta = \gamma_2 - \gamma_1$. 

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(iv) If $P_2^t \approx P_1^t$, then
\begin{equation}
\frac{dP_2^t}{dP_1^t} = e^{U_t},
\end{equation}
where
\begin{equation}
U_t = \langle \eta, X_t \rangle - \frac{1}{2} t \langle \eta, A \eta \rangle - t \langle \gamma_1, \eta \rangle
\end{equation}
with \( \eta \) satisfying \( A \eta = \gamma_2 - \gamma_1 \).

2. Scaled Poisson processes with drift. Suppose that both \( \{X_t\}, P_1 \) and \( \{X_t\}, P_2 \) are scaled Poisson processes with drift. That is, for \( j = 1, 2, \)
\begin{equation}
E^{P_j}[e^{izX_t}] = \exp \left[ t \left( b_j(e^{ia_j z} - 1) + i \gamma_{0j} z \right) \right], \quad z \in \mathbb{R},
\end{equation}
with \( b_j > 0, a_j \in \mathbb{R} \setminus \{0\}, \) and \( \gamma_{0j} \in \mathbb{R} \). Thus \( \nu_j = b_j \delta_{a_j} \). This is the case studied by Dvoretzky, Kiefer, and Wolfowitz [2] (1953). \( P_2^t \) and \( P_1^t \) are not mutually singular if and only if \( \gamma_{02} = \gamma_{01} \). Under the condition that \( \gamma_{02} = \gamma_{01} \), there are two cases.

Case 1: \( a_2 = a_1 \). In this case we have \( P_2^t \approx P_1^t \) and
\begin{equation}
P_2^t = (b_2/b_1)^{N_t} e^{-t(b_2-b_1)} P_1^t,
\end{equation}
where \( N_t = N_t(\omega) \) is the number of jumps of \( X_s(\omega) \) for \( s \leq t \).

Case 2: \( a_2 \neq a_1 \). In this case we have
\begin{equation}
(P_2^t)^{ac} = e^{t(b_1-b_2)} \Lambda_t P_1^t,
\end{equation}
where \( \Lambda_t = \{X_s - X_s- \neq a_1, a_2 \mbox{ for } s \in (0, t]\} \). Further we have \( (P_2^t)^{ac}(D) = e^{-t\nu_2} \) and \( (P_2^t)^{ac} = e^{-t\nu_2}Q^t \), where \( \{X_t\}, Q \) is a deterministic motion,
\[ Q(X_t = t\gamma_{02} \text{ for } t \geq 0) = 1. \]

3. Necessary conditions for (NS). Suppose that \( P_2^t \) and \( P_1^t \) are not mutually singular for \( 0 < t < \infty \). Then, one can prove from Theorem A that the following three cases are possible and that no other cases can arise:

Case 1: \( \nu_1(\mathbb{R}^d) < \infty \) and \( \nu_2(\mathbb{R}^d) < \infty \).

Case 2: \( \nu_1(\mathbb{R}^d) = \infty, \int_{|x| \leq 1} |x| \nu_1(dx) < \infty, \) and \( \nu_2(\mathbb{R}^d) = \infty, \int_{|x| \leq 1} |x| \nu_2(dx) < \infty. \)

Case 3: \( \int_{|x| \leq 1} |x| \nu_1(dx) = \infty \) and \( \int_{|x| \leq 1} |x| \nu_2(dx) = \infty. \)

Needless to say, these are not sufficient conditions for \( P_2^t \) and \( P_1^t \) not to be mutually singular.

4. Lévy processes with finite Lévy measures. Suppose that \( (A_j, \nu_j, \gamma_j), \) \( j = 1, 2, \) satisfy \( \nu_1(\mathbb{R}^d) < \infty, \nu_2(\mathbb{R}^d) < \infty, A_2 = A_1, \) and \( \gamma_{21} \in \mathcal{R}(A) \). Note that \( \gamma_{21} = \gamma_2 - \gamma_0. \)
where $\gamma_{0j}$, $j = 1, 2$, are respective drifts. We have
\begin{equation}
(P_t^1)^{ac}(D) = e^{-t\nu_2(\mathbb{R}^d)}.
\end{equation}
Thus, $P_t^2 \ll P_t^1$ if and only if $\nu_2 \ll \nu_1$.

5. Absolutely continuous change of Lévy measures. We start from one Lévy process $(\{X_t\}, P_1)$ on $\mathbb{R}^d$ with generating triplet $(A_1, \nu_1, \gamma_1)$. Suppose that we are given a measurable function $g(x)$ with values $-\infty \leq g(x) < \infty$ and a vector $\eta \in \mathbb{R}^d$. Assume that
\begin{equation}
\int_{\mathbb{R}^d} (e^{g(x)/2} - 1)^2 \nu_1(dx) < \infty.
\end{equation}
Define $(A_2, \nu_2, \gamma_2)$ by
\begin{equation}
A_2 = A_1, \quad \nu_2(dx) = e^{g(x)} \nu_1(dx), \quad \gamma_2 = \gamma_1 + \int_{|x| \leq 1} xd(\nu_2 - \nu_1) + A_1 \eta.
\end{equation}
Notice that (3.10) means that $k_{1/2}(\nu_1, \nu_2) < \infty$. Hence $\gamma_2$ is definable by Lemma 1.8.

The condition (3.10) is equivalent to the property that
\begin{equation}
\int_{|g| \leq 1} g^2 d\nu_1 + \int_{g > 1} e^g d\nu_1 + \int_{g < -1} d\nu_1 < \infty.
\end{equation}
It follows that
\begin{equation}
\int (1 \wedge |x|^2) \nu_2(dx) < \infty.
\end{equation}
A new Lévy process $(\{X_t\}, P_2)$ with generating triplet $(A_2, \nu_2, \gamma_2)$ is obtained in this way. We have $P_t^2 \ll P_t^1$ by Corollary 2.3 and $P_t^2 = e^{U_t} 1_{\Lambda_t} P_t^1$ in the notation of Theorem B. This procedure to get $(\{X_t\}, P_2)$ from $(\{X_t\}, P_1)$ is called density transformation in [8] and [9]. Esscher transformation (or exponential transformation) in [7] and [8] is a special case. Drift transformation, deletion of jumps, and truncation of the support of Lévy measure in [9] are also special cases.

References


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