SOME PROPERTIES OF EXPONENTIAL INTEGRALS OF LÉVY PROCESSES AND EXAMPLES

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Abstract. The improper stochastic integral $Z = \int_0^{\infty -} \exp(-X_{s-}) dY_s$ is studied, where $\{(X_t, Y_t), t \geqslant 0\}$ is a Lévy process on $\mathbb{R}^{1+d}$ with $\{X_t\}$ and $\{Y_t\}$ being $\mathbb{R}$-valued and $\mathbb{R}^d$-valued, respectively. The condition for existence and finiteness of Z is given and then the law $\mathcal{L}(Z)$ of Z is considered. Some sufficient conditions for $\mathcal{L}(Z)$ to be selfdecomposable and some sufficient conditions for $\mathcal{L}(Z)$ to be non-selfdecomposable but semi-selfdecomposable are given. Attention is paid to the case where $d = 1$, $\{X_t\}$ is a Poisson process, and $\{X_t\}$ and $\{Y_t\}$ are independent. An example of Z of type G with selfdecomposable mixing distribution is given.

1. Introduction

Let $\{(\xi_t, \eta_t), t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^2$. The generalized Ornstein-Uhlenbeck process $\{V_t, t \geqslant 0\}$ on $\mathbb{R}$ based on $\{(\xi_t, \eta_t), t \geqslant 0\}$ with initial condition V_0 is defined as

$$V_t = e^{-\xi_t} \left(V_0 + \int_0^t e^{\xi_{s-}} d\eta_s \right), \quad t \geqslant 0, \tag{1.1}$$

where V_0 is a random variable independent of $\{(\xi_t, \eta_t), t \geqslant 0\}$. This process has recently been well-studied by Carmona, Petit, and Yor [3], [4], Erickson and Maller [7], and Lindner and Maller [10].

Lindner and Maller [10] find that the generalized Ornstein-Uhlenbeck process $\{V_t, t \geqslant 0\}$ based on $\{(\xi_t, \eta_t), t \geqslant 0\}$ turns out to be a stationary process with a suitable choice of V_0 if and only if

$$P \left(\int_0^{\infty -} e^{-\xi_{s-}} dL_s \text{ exists and is finite} \right) = 1, \tag{1.2}$$

where

$$\int_0^{\infty-} e^{-\xi_{s-}} dL_s = \lim_{t\to\infty} \int_0^t e^{-\xi_{s-}} dL_s \tag{1.3}$$

and $\{(\xi_t, L_t), t \geqslant 0\}$ is a Lévy process on $\mathbb{R}^2$ defined by

$$L_t = \eta_t + \sum_{0<s\leqslant t} (e^{-(\xi_s - \xi_{s-})} - 1)(\eta_s - \eta_{s-}) - t a^{1,2}_{\xi,\eta} \tag{1.4}$$

with $\left(a^{j,k}_{\xi,\eta}\right)_{j,k=1,2}$ being the Gaussian covariance matrix in the Lévy–Khintchine triplet of the process $\{(\xi_t, \eta_t)\}$. Moreover, if the condition (1.2) is satisfied, then the choice of V_0 which makes $\{V_t\}$ stationary is unique in law and

$$\mathcal{L}(V_0) = \mathcal{L}\left(\int_0^{\infty-} e^{-\xi_{s-}} dL_s\right). \tag{1.5}$$

Here $\mathcal{L}$ stands for "the distribution of". If $\{\xi_t, t \geqslant 0\}$ and $\{\eta_t, t \geqslant 0\}$ are independent, then $P(L_t = \eta_t \text{ for all } t) = 1$.

Keeping in mind the results in the preceding paragraph, we study in this paper the exponential integral $\int_0^{\infty-} e^{-X_{s-}} dY_s$, where $\{(X_t, Y_t), t \geqslant 0\}$ is a Lévy process on $\mathbb{R}^{1+d}$ with $\{X_t\}$ and $\{Y_t\}$ being $\mathbb{R}$-valued and $\mathbb{R}^d$-valued, respectively. In Section 2 the existence conditions for this integral are given. They complement a theorem for $d = 1$ of Erickson and Maller [7]. Then, in Section 3, some properties of

$$\mu = \mathcal{L}\left(\int_0^{\infty-} e^{-X_{s-}} dY_s\right) \tag{1.6}$$

are studied. A sufficient condition for μ to be a selfdecomposable distribution on $\mathbb{R}^d$ is given as in Bertoin, Lindner, and Maller [2]. Further we give a sufficient condition for μ not to be selfdecomposable. Recall that, in the case where $X_t = t$, $t \geqslant 0$, and $\{Y_t\}$ is a Lévy process on $\mathbb{R}^d$, $\mathcal{L}\left(\int_0^{\infty-} e^{-s} dY_s\right)$ is always selfdecomposable if the integral exists and is finite (see e. g. [16], Section 17). In particular, we are interested in the case where $\{X_t\}$ and $\{Y_t\}$ are independent and $\{X_t\}$ is a Poisson process; we will give a sufficient condition for μ to be semi-selfdecomposable and not selfdecomposable and also a sufficient condition for μ to be selfdecomposable. In Section 4, we are concerned with μ of (1.6) when $\{X_t\}$ is a Brownian motion with positive drift on $\mathbb{R}$, $\{Y_t\}$ is a symmetric α-stable Lévy process on $\mathbb{R}$ with $0 < \alpha \leqslant 2$, and $\{X_t\}$ and $\{Y_t\}$ are independent. We will show that in this case μ gives a type G distribution with selfdecomposable mixing distribution, which is related to results in Maejima and Niiyama [12] and Aoyama, Maejima, and Rosiński [1].

2. Existence of exponential integrals of Lévy processes

Let $\{(X_t, Y_t), t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^{1+d}$, where $\{X_t\}$ is $\mathbb{R}$-valued and $\{Y_t\}$ is $\mathbb{R}^d$-valued. We keep this set-up throughout this section. Let (a_X, ν_X, γ_X) be the

Lévy-Khintchine triplet of the process $\{X_t\}$ in the sense that

$$Ee^{izX_t} = \exp\left[t\left(-\frac{1}{2}a_X z^2 + i\gamma_X z + \int_{\mathbb{R}\setminus\{0\}} (e^{izx} - 1 - izx1_{\{|x|\leqslant 1\}}(x))\nu_X(dx)\right)\right]$$

for $z \in \mathbb{R}$, where $a_X \geqslant 0$ and ν_X is the Lévy measure of $\{X_t\}$. Denote

$$h_X(x) = \gamma_X + \nu_X(\,(1,\infty)\,) + \int_1^x \nu_X(\,(y,\infty)\,)dy. \tag{2.1}$$

Let ν_Y be the Lévy measure of $\{Y_t\}$. The following result is a d-dimensional extension of Theorem 2 of Erickson and Maller [7].

Theorem 2.1. *Suppose that there is $c > 0$ such that $h_X(x) > 0$ for all $x \geqslant c$ and that $\{Y_t\}$ is not the zero process. Then*

$$P\left(\int_0^{\infty-} e^{-X_{s-}}dY_s \textit{ exists and is finite}\right) = 1 \tag{2.2}$$

if and only if

$$\lim_{t\to\infty} X_t = +\infty \textit{ a. s. and } \int_{|y|\geqslant e^c} \frac{\log|y|}{h_X(\log|y|)}\nu_Y(dy) < \infty, \tag{2.3}$$

where $|y|$ is the Euclidean norm of $y \in \mathbb{R}^d$.

Proof. First, for $d = 1$, this theorem is established in [7]. Second, for $j = 1,\dots,d$, the jth coordinate process $\{Y_t^{(j)}, t \geqslant 0\}$ is a Lévy process on $\mathbb{R}$ with Lévy measure $\nu_{Y^{(j)}}(B) = \int_{\mathbb{R}^d} 1_B(y_j)\nu_Y(dy)$ for any Borel set B in $\mathbb{R}$ satisfying $0 \notin B$, where $y = (y_1,\dots,y_d)$. Third, the property (2.2) is equivalent to

$$P\left(\int_0^{\infty-} e^{-X_{s-}}dY_s^{(j)} \text{ exists and is finite}\right) = 1 \quad \text{for } j = 1,\dots,d. \tag{2.4}$$

Next, we claim that the following (2.5) and (2.6) are equivalent:

$$\int_{|y|>M} \frac{\log|y|}{h_X(\log|y|)}\nu_Y(dy) < \infty \qquad \text{for some } M \geqslant e^c, \tag{2.5}$$

$$\int_{\{y\colon |y_j|>M\}} \frac{\log|y_j|}{h_X(\log|y_j|)}\nu_Y(dy) < \infty, \quad j = 1,\dots,d, \quad \text{for some } M \geqslant e^c. \tag{2.6}$$

Put $f(u) = \log u/h_X(\log u)$ for $u \geqslant e^c$. This $f(u)$ is not necessarily increasing for all $u \geqslant e^c$. We use the words *increasing* and *decreasing* in the wide sense allowing flatness. But $f(u)$ is increasing for sufficiently large u ($> M_0$, say), because, for $x > c$,

$$\frac{h_X(x)}{x} = \frac{h_X(c)}{x} + \frac{1}{x}\int_c^x n(y)dy$$

with $n(y) = \nu_X((y,\infty))$ and, with d/dx meaning the right derivative, we have

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}\int_c^x n(y)dy\right) &= \frac{1}{x^2}\left(-\int_c^x n(y)dy + xn(x)\right)\\ &= \frac{1}{x^2}\left(\int_c^x (n(x)-n(y))dy + cn(x)\right) < 0\end{aligned}$$

for sufficiently large x if $n(c) > 0$ (note that $\int_c^x (n(x)-n(y))dy$ is nonpositive and decreasing). Thus we see that (2.5) implies (2.6). Indeed, letting $M_1 = M \vee M_0$, we have

$$\int_{\{y:\,|y_j|>M_1\}} f(|y_j|)\nu_Y(dy) \leqslant \int_{\{y:\,|y_j|>M_1\}} f(|y|)\nu_Y(dy) \leqslant \int_{|y|>M_1} f(|y|)\nu_Y(dy) < \infty.$$

In order to show that (2.6) implies (2.5), let $g(x) = h_X(x)$ for $x \geqslant c$ and $= h_X(c)$ for $-\infty < x < c$. Then $g(x)$ is positive and increasing on $\mathbb{R}$. Assume (2.6). Let $M_1 = M \vee M_0$. Then, using the concavity of $\log(u+1)$ for $u \geqslant 0$, we have

$$\begin{aligned}\int_{|y|>M_1} f(|y|)\nu_Y(dy) &\leqslant \int_{|y|>M_1} f(|y_1|+\cdots+|y_d|)\nu_Y(dy)\\ &\leqslant \int_{|y|>M_1} \frac{\log(|y_1|+\cdots+|y_d|+1)}{h_X(\log(|y_1|+\cdots+|y_d|))}\nu_Y(dy)\\ &\leqslant \sum_{j=1}^d \int_{|y|>M_1} \frac{\log(|y_j|+1)}{h_X(\log(|y_1|+\cdots+|y_d|))}\nu_Y(dy),\\ &= \sum_{j=1}^d \int_{|y|>M_1} \frac{\log(|y_j|+1)}{g(\log(|y_1|+\cdots+|y_d|))}\nu_Y(dy)\\ &\leqslant \sum_{j=1}^d \int_{|y|>M_1} \frac{\log(|y_j|+1)}{g(\log(|y_j|))}\nu_\eta(dy)\\ &\leqslant \sum_{j=1}^d \left(\int_{|y_j|>M_1} \frac{\log(|y_j|+1)}{g(\log(|y_j|))}\nu_Y(dy) + \int_{|y_j|\leqslant M_1,\,|y|>M_1} \frac{\log(|y_j|+1)}{g(\log(|y_j|))}\nu_Y(dy)\right).\end{aligned}$$

The first integral in each summand is finite due to (2.6) and the second integral is also finite because the integrand is bounded. This finishes the proof of equivalence of (2.5) and (2.6).

Now assume that (2.3) holds. Then (2.6) holds. Hence, by the theorem for $d=1$, $\int_0^{\infty-} e^{-X_{s-}}dY_s^{(j)}$ exists and is finite a. s. for all j such that $\{Y_t^{(j)}\}$ is not the zero process. For j such that $\{Y_t^{(j)}\}$ is the zero process, we have $\int_0^{\infty-} e^{-X_{s-}}dY_s^{(j)} = 0$. Hence (2.4) holds, that is, (2.2) holds.

Conversely, assume that (2.2) holds. Let

$$I_j = \int_{\{y:\,|y_j|\geqslant e^c\}} \frac{\log|y_j|}{h_X(\log|y_j|)}\nu_Y(dy).$$

Since $\{Y_t\}$ is not the zero process, $\{Y_t^{(j)}\}$ is not the zero process for some j. Hence, by the theorem for $d = 1$, $\lim_{t\to\infty} X_t = +\infty$ a. s. and $I_j < \infty$ for such j. For j such that $\{Y_t^{(j)}\}$ is the zero process, $\nu_{Y^{(j)}} = 0$ and $I_j = 0$. Hence we have (2.6) and thus (2.3) holds due to the equivalence of (2.5) and (2.6). $\square$

Remark 2.2. (i) Suppose that $\{X_t\}$ satisfies $0 < EX_1 < \infty$. Then $\lim_{t\to\infty} X_t = +\infty$ a. s. and $h_X(x)$ is positive and bounded for large x. Thus (2.2) holds if and only if

$$\int_{\mathbb{R}^d} \log^+ |y|\, \nu_Y(dy) < \infty. \tag{2.7}$$

Here $\log^+ u = 0 \vee \log u$. For $d = 1$ this is mentioned in the comments following Theorem 2 of [7].

(ii) As is pointed out in Theorem 5.8 of Sato [17], $\lim_{t\to\infty} X_t = +\infty$ a. s. if and only if one of the following (a) and (b) holds:
(a) $E(X_1 \wedge 0) > -\infty$ and $0 < EX_1 \leqslant +\infty$;
(b) $E(X_1 \wedge 0) = -\infty$, $E(X_1 \vee 0) = +\infty$, and

$$\int_{(-\infty,-2)} |x| \left(\int_1^{|x|} \nu_X(\,(y,\infty)\,)dy \right)^{-1} \nu_X(dx) < \infty. \tag{2.8}$$

In other words, $\lim_{t\to\infty} X_t = +\infty$ a. s. if and only if one of the following (a′) and (b′) holds:
(a′) $E|X_1| < \infty$ and $EX_1 > 0$;
(b′) $\int_1^\infty \nu_X(\,(y,\infty)\,)dy = \infty$ and (2.8) holds.
See also Doney and Maller [5].

(iii) If $\lim_{t\to\infty} X_t = +\infty$ a. s., then $h_X(x) > 0$ for all large x, as is explained in [7] after their Theorem 2.

When $\{X_t\}$ and $\{Y_t\}$ are independent, the result in Remark 2.2 (i) can be extended to more general exponential integrals of Lévy processes.

Theorem 2.3. *Suppose that $\{X_t\}$ and $\{Y_t\}$ are independent and that $0 < EX_1 < \infty$. Let $\alpha > 0$. Then*

$$P\left(\int_0^{\infty-} e^{-(X_{s-})^\alpha} dY_s \text{ exists and is finite} \right) = 1 \tag{2.9}$$

if and only if

$$\int_{\mathbb{R}^d} (\log^+ |y|)^{1/\alpha} \nu_Y(dy) < \infty. \tag{2.10}$$

We use the following result, which is a part of Proposition 4.3 of [19].

Proposition 2.4. *Let f be a locally square-integrable nonrandom function on $[0,\infty)$ such that there are positive constants α, c_1, and c_2 satisfying*

$$e^{-c_2 s^\alpha} \leqslant f(s) \leqslant e^{-c_1 s^\alpha} \quad \text{for all large } s.$$

Then

$$P\left(\int_0^{\infty-} f(s) dY_s \text{ exists and is finite} \right) = 1$$

if and only if (2.10) *holds.*

Proof of Theorem 2.3. Let $E[X_1] = b$. By assumption, $0 < b < \infty$. By the law of large numbers for Lévy processes (Theorem 36.5 of [16]), we have $\lim_{t\to\infty} X_t/t = b$ a. s. Hence

$$P(b/2 < \xi_t/t < 2b \text{ for all large } t) = 1.$$

Conditioned by the process $\{X_t\}$, the integral $\int_0^t e^{-(X_{s-})^\alpha} dY_s$ can be considered as that with X_s, $s \geqslant 0$, frozen while Y_s, $s \geqslant 0$, maintains the same randomness. This is because the two processes are independent. Hence we can apply Proposition 2.4. Thus, if (2.10) holds, then

$$\begin{aligned} &P\left(\int_0^{\infty-} e^{-(X_{s-})^\alpha} dY_s \text{ exists and is finite}\right) \\ &\qquad = E\left[P\left(\int_0^{\infty-} e^{-(X_{s-})^\alpha} dY_s \text{ exists and is finite} \,\middle|\, \{X_t\}\right)\right] = 1. \end{aligned}$$

Conversely, if (2.10) does not hold, then

$$P\left(\int_0^{\infty-} e^{-(X_{s-})^\alpha} dY_s \text{ exists and is finite}\right) = 0.$$

Indeed, in the situation of Proposition 2.4, we have, by Kolmogorov's zero-one law,

$$P\left(\int_0^{\infty-} f(s) dY_s \text{ exists and is finite}\right) = 0$$

if and only if (2.10) does not hold. □

3. Properties of the laws of exponential integrals of Lévy processes.

Let μ be a distribution on $\mathbb{R}^d$. Denote by $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$, the characteristic function of μ. We call μ selfdecomposable if, for every $b \in (0,1)$, there is a distribution ρ_b on $\mathbb{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z). \tag{3.1}$$

If μ is selfdecomposable, then μ is infinitely divisible and ρ_b is uniquely determined and infinitely divisible. If, for a fixed $b \in (0,1)$, there is an infinitely divisible distribution ρ_b on $\mathbb{R}^d$ satisfying (3.1), then μ is called b-semi-selfdecomposable, or of class $L_0(b, \mathbb{R}^d)$. If μ is b-semi-selfdecomposable, then μ is infinitely divisible and ρ_b is uniquely determined. If μ is b-semi-selfdecomposable and ρ_b is of class $L_0(b, \mathbb{R}^d)$, then μ is called of class $L_1(b, \mathbb{R}^d)$. These "semi"-concepts were introduced by Maejima and Naito [11].

We start with a sufficient condition for selfdecomposability of the laws of exponential integrals of Lévy processes.

Theorem 3.1. *Suppose that $\{(X_t, Y_t), t \geqslant 0\}$ is a Lévy process on $\mathbb{R}^{1+d}$, where $\{X_t\}$ is $\mathbb{R}$-valued and $\{Y_t\}$ is $\mathbb{R}^d$-valued. Suppose in addition that $\{X_t\}$ does not have* *positive jumps and* $0 < EX_1 < +\infty$ *and that*

$$\int_{\mathbb{R}^d} \log^+ |y| \nu_Y(dy) < \infty \tag{3.2}$$

for the Lévy measure ν_Y *of* $\{Y_t\}$*. Let*

$$\mu = \mathcal{L}\left(\int_0^{\infty-} e^{-X_{s-}} dY_s \right). \tag{3.3}$$

Then μ *is selfdecomposable.*

When $d = 1$ and $Y_t = t$, the assertion is found in [9]. When $d = 1$, the assertion of this theorem is found in the paper [2] with a key idea of the proof. This fact was informed personally by Alex Lindner to the second author of the present paper when he was visiting Munich in November, 2005, while the paper [2] was in preparation. For $d \geqslant 2$ we do not need a new idea but, for completeness, we give a proof of it here.

Proof of Theorem 3.1. If $\{Y_t\}$ is the zero process, then the theorem is trivial. Hence we assume that $\{Y_t\}$ is not the zero process. Under the assumption that $\{X_t\}$ does not have positive jumps, we have that $\lim_{t\to\infty} X_t = +\infty$ a. s. if and only if $0 < EX_1 < +\infty$. Thus the integral $Z = \int_0^{\infty-} e^{-X_{s-}} dY_s$ exists and is finite a. s. by virtue of Theorem 2.1 and Remark 2.2 (i). Let $c > 0$, and define

$$T_c = \inf\{t \colon X_t = c\}.$$

Since we are assuming that X_t does not have positive jumps and that $0 < EX_1 < +\infty$, we have $T_c < \infty$ and $X(T_c) = c$ a. s. Then we have

$$Z = \int_0^{\infty-} e^{-X_{s-}} dY_s = \int_0^{T_c} e^{-X_{s-}} dY_s + \int_{T_c}^{\infty-} e^{-X_{s-}} dY_s.$$

Denote by U_c and V_c the first and second integral of the last member. We have

$$V_c = \int_{T_c}^{\infty-} e^{-X(s-)+X(T_c)-X(T_c)} dY_s = e^{-c} Z_c,$$

where

$$Z_c = \int_{T_c}^{\infty-} e^{-X(s-)+X(T_c)} dY_s = \int_0^{\infty-} e^{-(X(T_c+s-)-X(T_c))} d(Y(T_c+s)-Y(T_c)).$$

Since T_c is a stopping time for the process $\{(X_s, Y_s), s \geqslant 0\}$, we see that $\{(X(T_c + s) - X(T_c), Y(T_c + s) - Y(T_c)), s \geqslant 0\}$ and $\{(X_s, Y_s), 0 \leqslant s \leqslant T_c\}$ are independent and the former process is identical in law with $\{(X_s, Y_s), s \geqslant 0\}$ (see Theorem 40.10 of [16]). Thus Z_c and U_c are independent and $\mathcal{L}(Z_c) = \mathcal{L}(Z)$. Since c is arbitrary, it follows that the law of Z is selfdecomposable. □

We turn our attention to the case where $\{X_t\}$ is a Poisson process and $\{X_t\}$ and $\{Y_t\}$ are independent. The suggestion of studying this case was personally given by Jan Rosiński to the authors. In this case we will show that the law μ of the exponential integral can be selfdecomposable or non-selfdecomposable, depending on the choice of $\{Y_t\}$. A measure ν on $\mathbb{R}^d$ is called discrete if it is concentrated on some countable set C, that is, $\nu(\mathbb{R}^d \setminus C) = 0$.

Theorem 3.2. *Suppose that $\{N_t, t \geqslant 0\}$ is a Poisson process, $\{Y_t\}$ is a Lévy process on $\mathbb{R}^d$, and $\{N_t\}$ and $\{Y_t\}$ are independent. Suppose that (3.2) holds. Let*

$$\mu = \mathcal{L}\left(\int_0^{\infty-} e^{-N_{s-}} dY_s\right). \tag{3.4}$$

Then the following statements are true.

(i) *The law μ is infinitely divisible and, furthermore, e^{-1}-semi-selfdecomposable.*

(ii) *Suppose that either $\{Y_t\}$ is a strictly α-stable Lévy process on $\mathbb{R}^d$, $d \geqslant 1$, with $0 < \alpha \leqslant 2$ or $\{Y_t\}$ is a Brownian motion with drift with $d = 1$. Then, μ is selfdecomposable and of class $L_1(e^{-1}, \mathbb{R}^d)$.*

(iii) *Suppose that $d = 1$ and $\{Y_t\}$ is integer-valued, not identically zero. Let*

$$D = \begin{cases} (0, \infty) & \text{if } \{Y_t\} \text{ is increasing,} \\ (-\infty, 0) & \text{if } \{Y_t\} \text{ is decreasing,} \\ \mathbb{R} & \text{if } \{Y_t\} \text{ is neither increasing nor decreasing.} \end{cases}$$

Then μ is not selfdecomposable and, furthermore, the Lévy measure ν_μ of μ is discrete and the set of points with positive ν_μ-measure is dense in D.

It is noteworthy that a seemingly pathological Lévy measure appears in a natural way in the assertion (iii). In relation to the infinite divisibility in (i), we recall that $\int_0^{\infty-} \exp(-N_{s-} - cs)ds$ does not have an infinitely divisible law if $c > 0$. This is Samorodnitsky's remark mentioned in [9]. The integral $\int_0^{\infty-} \exp(-N_{s-})ds$ is a special case of (ii) with $\alpha = 1$.

Proof of Theorem 3.2. (i) Let $Z = \int_0^{\infty-} e^{-N_{s-}} dY_s$. If $\{Y_t\}$ is the zero process, then $Z = 0$. If $\{Y_t\}$ is not the zero process, then existence and finiteness of Z follows from Theorem 2.1. Let $T_n = \inf\{s \geqslant 0 \colon N_s = n\}$. Clearly T_n is finite and tends to infinity as $n \to \infty$ a. s. We have

$$Z = \sum_{n=0}^{\infty} \int_{T_n}^{T_{n+1}} e^{-N_{s-}} dY_s = \sum_{n=0}^{\infty} e^{-n}(Y(T_{n+1}) - Y(T_n)).$$

For each n, T_n is a stopping time for $\{(N_s, Y_s) \colon s \geqslant 0\}$. Hence $\{(N(T_n + s) - N(T_n), Y(T_n + s) - Y(T_n)), s \geqslant 0\}$ and $\{(N_s, Y_s), 0 \leqslant s \leqslant T_n\}$ are independent and the former process is identical in law with $\{(N_s, Y_s), s \geqslant 0\}$. It follows that the family $\{Y(T_{n+1}) - Y(T_n), n = 0, 1, 2, \ldots\}$ is independent and identically distributed. Thus, denoting $W_n = Y(T_{n+1}) - Y(T_n)$, we have representation

$$Z = \sum_{n=0}^{\infty} e^{-n} W_n, \tag{3.5}$$

where $W_0, W_1 \ldots.$ are independent and identically distributed and $W_n \stackrel{\mathrm{d}}{=} Y(T_1)$ ($\stackrel{\mathrm{d}}{=}$ stands for "has the same law as"). Consequently we have

$$Z = W_0 + e^{-1} Z', \tag{3.6}$$

where W_0 and Z' are independent and $Z' \overset{\mathrm{d}}{=} Z$. The distribution of W_0 is infinitely divisible, since $W_0 = Y(T_1) \overset{\mathrm{d}}{=} U_1$, where $\{U_s\}$ is a Lévy process given by subordination of $\{Y_s\}$ by a gamma process. Here we use our assumption of independence of $\{N_t\}$ and $\{Y_t\}$. Thus μ is e^{-1}-semi-selfdecomposable and hence infinitely divisible. An alternative proof of the infinite divisibility of μ is to look at the representation (3.5) and to use that $\mathcal{L}(Y(T_1))$ is infinitely divisible.

(ii) Use the representation (3.5) with $W_n \overset{\mathrm{d}}{=} U_1$, where we obtain a Lévy process $\{U_s\}$ by subordination of $\{Y_s\}$ by a gamma process. Since gamma distributions are selfdecomposable, the results of Sato [18] on inheritance of selfdecomposability in subordination guarantee that $\mathcal{L}(U_1)$ is selfdecomposable under our assumption on $\{Y_s\}$. Hence μ is selfdecomposable, as selfdecomposability is preserved under convolution and convergence. Further, since selfdecomposability implies b-semi-selfdecomposability for each b, (3.6) shows that μ is of class $L_1(e^{-1}, \mathbb{R}^d)$.

(iii) The process $\{Y_t\}$ is a compound Poisson process on $\mathbb{R}$ with ν_Y concentrated on the integers (see Corollary 24.6 of [16]). Let us consider the Lévy measure $\nu^{(0)}$ of $Y(T_1)$. Let $a > 0$ be the parameter of the Poisson process $\{N_t\}$. As in the proofs of (i) and (ii), $Y(T_1) \overset{\mathrm{d}}{=} U_1$, where $\{U_s\}$ is given by subordination of $\{Y_s\}$, by a gamma process which has Lévy measure $x^{-1}e^{-ax}dx$. Hence, using Theorem 30.1 of [16], we see that

$$\nu^{(0)}(B) = \int_0^\infty P(Y_s \in B) s^{-1} e^{-as} ds$$

for any Borel set B in $\mathbb{R}$. Thus $\nu^{(0)}(\mathbb{R} \setminus \mathbb{Z}) = 0$.

Suppose that $\{Y_t\}$ is not a decreasing process. Then some positive integer has positive $\nu^{(0)}$-measure. Denote by p the minimum of such positive integers. Since $\{Y_t\}$ is compound Poisson, $P(Y_s = kp) > 0$ for any $s > 0$ for $k = 1, 2, \ldots$. Hence $\nu^{(0)}(\{kp\}) > 0$ for $k = 1, 2, \ldots$. Therefore, for each nonnegative integer n, the Lévy measure $\nu^{(n)}$ of $e^{-n}Y(T_1)$ satisfies $\nu^{(n)}(\{e^{-n}kp\}) > 0$ for $k = 1, 2, \ldots$. Clearly, $\nu^{(n)}$ is also discrete. The representation (3.5) shows that

$$\nu_\mu = \sum_{n=0}^{\infty} \nu^{(n)}.$$

Hence, ν_μ is discrete and

$$\nu_\mu(\{e^{-n}kp\}) > 0 \quad \text{for all } n = 0, 1, 2, \ldots \text{ and } k = 1, 2, \ldots .$$

Thus the points in $(0, \infty)$ of positive ν_μ-measure are dense in $(0, \infty)$.

Similarly, if $\{Y_t\}$ is not an increasing process, then the points in $(-\infty, 0)$ of positive ν_μ-measure are dense in $(-\infty, 0)$. $\square$

The following remarks give information on continuity properties of the law μ. A distribution on $\mathbb{R}^d$ is called nondegenerate if its support is not contained in any affine subspace of dimension $d - 1$.

Remark 3.3. (i) Any nondegenerate selfdecomposable distribution on $\mathbb{R}^d$ for $d \geqslant 1$ is absolutely continuous (with respect to Lebesgue measure on $\mathbb{R}^d$) although, for $d \geqslant 2$, its Lévy measure is not necessarily absolutely continuous. This is proved by Sato [15] (see also Theorem 27.13 of [16]).

(ii) Nondegenerate semi-selfdecomposable distributions on $\mathbb{R}^d$ for $d \geqslant 1$ are absolutely continuous or continuous singular, as Wolfe [20] proves (see also Theorem 27.15 of [16]).

4. An example of type G random variable

In Maejima and Niiyama [12], an improper integral

$$Z = \int_0^{\infty-} e^{-(B_s+\lambda s)} dS_s \tag{4.1}$$

was studied, in relation to a stationary solution of the stochastic differential equation

$$dZ_t = -\lambda Z_t dt + Z_{t-} dB_t + dS_t, \quad t \geqslant 0,$$

where $\{B_t, t \geqslant 0\}$ is a standard Brownian motion on $\mathbb{R}$, $\lambda > 0$, and $\{S_t, t \geqslant 0\}$ is a symmetric α-stable Lévy process with $0 < \alpha \leqslant 2$ on $\mathbb{R}$, independent of $\{B_t\}$. They showed that Z is *of type* G in the sense that Z is a variance mixture of a standard normal random variable by some infinitely divisible distribution. Namely, Z is of type G if

$$Z \stackrel{\mathrm{d}}{=} V^{1/2} W$$

for some nonnegative infinitely divisible random variable V and a standard normal random variable W independent of each other. Equivalently, Z is of type G if and only if $Z \stackrel{\mathrm{d}}{=} U_1$, where $\{U_t, t \geqslant 0\}$ is given by subordination of a standard Brownian motion. If Z is of type G, then $\mathcal{L}(V)$ is uniquely determined by $\mathcal{L}(Z)$ (Lemma 3.1 of [18]).

The Z in (4.1) is a special case of those exponential integrals of Lévy processes which we are dealing with. Thus Theorem 3.1 says that the law of Z is selfdecomposable. But the class of type G distributions (the laws of type G random variables) is neither larger nor smaller than the class of symmetric selfdecomposable distributions. Although the proof that Z is of type G is found in [12], the research report is not well distributed. Hence we give their proof below for readers. We will show that the law of Z belongs to a special subclass of selfdecomposable distributions.

Theorem 4.1. *Under the assumptions on $\{B_t\}$ and $\{S_t\}$ stated above, Z in (4.1) is of type G and furthermore the mixing distribution for variance, $\mathcal{L}(V)$, is not only infinitely divisible but also selfdecomposable.*

Proof. It is known (Proposition 4.4.4 of Dufresne [6]) that for any $a \in \mathbb{R} \setminus \{0\}$, $b > 0$,

$$\int_0^{\infty} e^{aB_s - bs} ds \stackrel{\mathrm{d}}{=} 2 \left(a^2 \Gamma_{2ba^{-2}}\right)^{-1},$$

where Γ_γ is the gamma random variable with parameter $\gamma > 0$, namely, $P(\Gamma_\gamma \in B) = \Gamma(\gamma)^{-1} \int_{B\cap(0,\infty)} x^{\gamma-1} e^{-x} dx$. The law of the reciprocal of gamma random variable is infinitely divisible and, furthermore, selfdecomposable (Halgreen [8]). We have

$$E\left[e^{izZ}\right] = E\left[\exp\left(iz\int_0^{\infty-} e^{-(B_s+\lambda s)}dS_s\right)\right]$$
$$= E\left[E\left[\exp\left(iz\int_0^{\infty-} e^{-(B_s+\lambda s)}dS_s\right)\Bigg|\ \{B_s\}\right]\right],$$

We have $Ee^{izS_t} = \exp(-ct|z|^\alpha)$ with some $c > 0$. For any nonrandom measurable function $f(s)$ satisfying $\int_0^\infty |f(s)|^\alpha ds < \infty$, we have

$$E\left[\exp\left(iz\int_0^{\infty-} f(s)dS_s\right)\right] = \exp\left(-c|z|^\alpha\int_0^\infty |f(s)|^\alpha ds\right)$$

(see, e. g. Samorodnitsky and Taqqu [14]). Hence

$$E\left[e^{izZ}\right] = E\left[\exp\left(-c|z|^\alpha\int_0^\infty e^{-\alpha B_s-\alpha\lambda s}ds\right)\right]$$
$$= E\left[\exp\left(-c|z|^\alpha 2\left(\alpha^2\Gamma_{2\alpha^{-1}\lambda}\right)^{-1}\right)\right].$$

If we put

$$H(dx) = P\left(2c\left(\alpha^2\Gamma_{2\alpha^{-1}\lambda}\right)^{-1}\in dx\right),$$

then

$$E[e^{izZ}] = \int_0^\infty e^{-u|z|^\alpha}H(du).$$

This H is the distribution of a positive infinitely divisible (actually selfdecomposable) random variable. This shows that Z is a mixture of a symmetric α-stable random variable S with $Ee^{izS} = e^{-|z|^\alpha}$ in the sense that

$$Z \stackrel{\mathrm{d}}{=} \Gamma^{-1/\alpha}S, \tag{4.2}$$

where Γ and S are independent and Γ is a gamma random variable with $\mathcal{L}(\Gamma^{-1}) = H$, that is, $\Gamma = (2c)^{-1}\alpha^2\Gamma_{2\alpha^{-1}\lambda}$. To see that Z is of type G, we need to rewrite (4.2) as

$$Z \stackrel{\mathrm{d}}{=} \Gamma^{-1/\alpha}S \stackrel{\mathrm{d}}{=} V^{1/2}W,$$

for some infinitely divisible random variable $V > 0$ independent of a standard normal random variable W. Let $S^+_{\alpha/2}$ be a positive strictly $(\alpha/2)$-stable random variable such that

$$E\left[\exp(-uS^+_{\alpha/2})\right] = \exp\left(-(2u)^{\alpha/2}\right), \quad u \geqslant 0$$

and Γ, W, and $S^+_{\alpha/2}$ are independent. Then

$$S \stackrel{\mathrm{d}}{=} (S^+_{\alpha/2})^{1/2}W,$$

and hence S is of type G. Let

$$V = \Gamma^{-2/\alpha}S^+_{\alpha/2}.$$

Then

$$V^{1/2}W = (\Gamma^{-2/\alpha}S^+_{\alpha/2})^{1/2}W = \Gamma^{-1/\alpha}(S^+_{\alpha/2})^{1/2}W \stackrel{\mathrm{d}}{=} \Gamma^{-1/\alpha}S \stackrel{\mathrm{d}}{=} Z.$$

Using a positive strictly $(\alpha/2)$-stable Lévy process $\{S^+_{\alpha/2}(t), t \geqslant 0\}$ independent of Γ with $\mathcal{L}(S^+_{\alpha/2}(1)) = S^+_{\alpha/2}$, we see that

$$V \stackrel{\mathrm{d}}{=} S^+_{\alpha/2}(\Gamma^{-1}).$$

Since Γ^{-1} is selfdecomposable, V is also selfdecomposable due to the inheritance of selfdecomposability in subordination of strictly stable Lévy processes (see [18]). Therefore Z is of type G with $\mathcal{L}(V)$ being selfdecomposable. Also, the selfdecomposability of Z again follows. $\square$

In their recent paper [1], Aoyama, Maejima, and Rosiński have introduced a new strict subclass (called $M(\mathbb{R}^d)$) of the intersection of the class of type G distributions and the class of selfdecomposable distributions on $\mathbb{R}^d$ (see Maejima and Rosiński [13] for the definition of type G distributions on $\mathbb{R}^d$ for general d). If we write the polar decomposition of the Lévy measure ν by

$$\nu(B) = \int_K \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),$$

where K is the unit sphere $\{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$ and λ is a probability measure on K, then the element of $M(\mathbb{R}^d)$ is characterized as a symmetric infinitely divisible distribution such that

$$\nu_\xi(dr) = g_\xi(r^2)r^{-1}dr$$

with $g_\xi(u)$ being completely monotone as a function of $u \in (0,\infty)$ and measurable with respect to ξ. Recall that if we write $\nu_\xi(dr) = g_\xi(r^2)dr$ instead, this gives a characterization of type G distributions on $\mathbb{R}^d$ ([13]). In [1] it is shown that

$$\{\text{type } G \text{ distributions on } \mathbb{R} \text{ with selfdecomposable mixing distributions}\} \subsetneqq M(\mathbb{R}).$$

Now, by Theorem 4.1 combined with the observation above, we see that $\mathcal{L}(Z)$ in (4.1) belongs to $M(\mathbb{R})$. It is of interest as a concrete example of random variable whose distribution belongs to $M(\mathbb{R})$.

We end the paper with a remark that, by Preposition 3.2 of [4], if $\alpha = 2$, our $\mathcal{L}(Z)$ is also Pearson type IV distribution of parameters λ and 0.

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References

[1] T. Aoyama, M. Maejima and J. Rosiński (2006) : A subclass of type G selfdecomposable distributions, preprint.

[2] J. Bertoin, A. Lindner and R. Maller (2006) : On continuity properties of the law of integrals of Lévy processes, preprint.

[3] Ph. Carmona, F. Petit and M. Yor (1997) : On the distribution and asymptotic results for exponential functionals of Lévy processes, in *Exponential Functionals and Principal Values Related to Brownian Motion* (ed. M. Yor), Biblioteca de la Revista Matemática Iberoamericana, pp. 73–126.

[4] Ph. Carmona, F. Petit and M. Yor (2001) : Exponential functionals of Lévy processes, in *Lévy Processes: Theory and Applications,* eds. O.E. Barndorff-Nielsen, T. Mikosh and S.I. Resnick, Birkhäuser, pp. 41–55.

[5] R. Doney and R.A. Maller (2002) : Stability and attraction to normality for Lévy processes at zero and at infinity, *J. Theoret. Prob.* **15**, 751–792.

[6] D. Dufresne (1990) : The distribution of a perpetuity, with applications to risk theory and pension funding, *Scand. Actuarial J.* **1990**, 39-79.

[7] K.B. Erickson and R. Maller (2005) : Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals, *Lecture Note in Math.*, Springer, **1857**, 70–94.

[8] C. Halgreen (1979) : Self-decomposability of the ggeneralized inverse Gaussian and hyperbolic distributions, *Z. Wahrsch. Verw. Gebiete* **47**, 13–17.

[9] C. Klüppelberg, A. Lindner and R. Maller (2006) : Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models, in *The Shiryaev Festschrift: From Stochastic Calculus to Mathematical Finance*, eds. Yu. Kabanov, R. Liptser and J. Stoyanov, Springer, Berlin, pp. 393–419,

[10] A. Lindner and R. Maller (2005) : Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes, *Stoch. Proc. Appl.* **115**, 1701–1722.

[11] M. Maejima and Y. Naito (1998) : Semi-selfdecomposable distributions and a new class of limit theorems, *Probab. Theory Relat. Fields* **112**, 12–31.

[12] M. Maejima and Y. Niiyama (2005) : The generalized Langevin equation and an example of type G distributions, *ISM Cooperate Research Report* **175**, 126–129.

[13] M. Maejima and J. Rosiński (2002) : Type G distributions on $\mathbb{R}^d$, *J. Theoret. Prob.* **15**, 323–341.

[14] G. Samorodnitsky and M.S. Taqqu (1994) : *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York.

[15] K. Sato (1982) : Absolute continuity of multivariate distributions of class L, *J. Multivar. Anal.* **12**, 89–94.

[16] K. Sato (1999) : *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge.

[17] K. Sato (2001) : Basic results on Lévy processes, in *Lévy Processes: Theory and Applications*, eds. O.E. Barndorff-Nielsen, T. Mikosch and S.I. Resnick, Birkhäuser, Boston, pp. 3–37.

[18] K. Sato (2001) : Subordination and self-decomposability, *Statist. Probab. Lett.* **54**, 317–324.

[19] K. Sato (2005) : Monotonicity and non-monotonicity of domains of stochastic integral operators, preprint.

[20] S.J. Wolfe (1983) : Continuity properties of decomposable probability measures on Euclidean spaces, *J. Multivar. Anal.* **13**, 534–538.