Infinite Divisibility for Stochastic Processes and Time Change

Ole E. Barndorff-Nielsen[1], **Makoto Maejima**[2] **and Ken-iti Sato**[3] [4]

Running head: Infinite Divisibility for Stochastic Processes and Time Change

[1]Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark.

[2]Department of Mathematics, Keio University, Hiyoshi, Yokohama, 223-8522 Japan.

[3]Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya, 468-0074 Japan.

[4]To whom correspondence should be addressed. E-mail: ken-iti.sato@nifty.ne.jp

Infinite Divisibility for Stochastic Processes and Time Change

Ole E. Barndorff-Nielsen[1], **Makoto Maejima**[2] **and Ken-iti Sato**[3 4]

General results concerning infinite divisibility, selfdecomposability, and the class L_m property as properties of stochastic processes are presented. A new concept called temporal selfdecomposability of stochastic processes is introduced. Lévy processes, additive processes, selfsimilar processes, and stationary processes of Ornstein-Uhlenbeck type are studied in relation to these concepts. Further, time change of stochastic processes is studied, where chronometers (stochastic processes that serve to change time) and base processes (processes to be time-changed) are independent but do not, in general, have independent increments. Conditions for inheritance of infinite divisibility and selfdecomposability under time change are given.

KEY WORDS: Infinite divisibility; selfdecomposability; temporal selfdecomposability; class L_m property; time change; chronometer.

1. Introduction

The first purpose of the present paper is to study infinite divisibility, selfdecomposability, and the class L_m property as properties of stochastic processes. Several relations between the various concepts and some basic properties are given in Section 3. These concepts are studied especially for stationary processes of Ornstein-Uhlenbeck type in Section 4. A new concept of temporal selfdecomposability of stochastic processes is introduced in Section 5. This concept is wider than the concept of Lévy processes but, under a slight restriction, narrower than that of infinitely divisible processes. We show that there exists a temporally selfdecomposable non-Lévy process whose one-dimensional marginals coincide with those of a Lévy process.

The second purpose of the paper is to discuss time change. Time change of stochastic processes is a topic of considerable current interest. This is especially so for cases where the stochastic process, that is being time-changed, is a Lévy process. We shall generally refer to stochastic processes that serve to change time by the term *chronometers* and processes that are to be time-changed as *base processes.* (Bochner's) subordination, i.e. where the chronometer is a Lévy process independent of the base process and the base process is a Lévy process or, more generally, a time-homogeneous Markov process, is a classical area, initiated by Bochner[(11,12)];

[1]Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark.

[2]Department of Mathematics, Keio University, Hiyoshi, Yokohama, 223-8522 Japan.

[3]Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya, 468-0074 Japan.

[4]To whom correspondence should be addressed. E-mail: ken-iti.sato@nifty.ne.jp

some recent references are Bertoin[9,10], Sato[34], and Barndorff-Nielsen, Pedersen and Sato[4]. There is a wide range of Lévy processes, obtained by subordination of Brownian motion, which are of interest as models in mathematical finance. See, for instance, Eberlein[17], Geman, Madan and Yor[20], Carr, Geman, Madan and Yor[13], Eberlein and Prause[18], and references given there. Time-changed Lévy processes where the chronometers are more general than subordinators, being for instance continuous and in the form of the integral of some volatility process, play a major role in modelling in finance, see for instance Barndorff-Nielsen and Shephard[5], Barndorff-Nielsen, Nicolato and Shephard[3], Carr, Geman, Madan and Yor[14], Cont and Tankov[16], Barndorff-Nielsen and Shephard[7], and references given there. In most of the work referred to above the chronometer is assumed to be independent of the base process. Furthermore, the latter process is a Lévy process and the chronometer is an infinitely divisible process. We discuss time change of stochastic process in the last two sections. Section 6 is for a study of chronometers. Section 7 contains a main result on inheritance of infinite divisibility under time change when base processes are Lévy processes.

Note that there is another type of time change which is frequently used in the theory of Markov processes. In this type of time change, the base process is a time-homogeneous strong Markov process and the chronometer is determined by the base process as the inverse of a nonnegative continuous additive functional of the base process. This situation is quite different from that of the subordination, where the independence of chronometer and base process is essential. A well-known example is construction of all one-dimensional regular diffusion processes from Brownian motion by scale change and time change; see Itô and McKean[22]. The time change we work on in this paper does not include this type.

Finally, we give reference to some other recent work on time change that considers aspects different from those of the present paper. It is a question of some special interest to what extent information on the chronometer can be obtained from observing the time-changed process only. This question is considered for Brownian subordination in Geman, Madan and Yor[21] and their work has been extended by Winkel[37] to time change of Lévy processes with more general chronometers. For some discussions of time change in quantum physics and in turbulence see Chung and Zambrini[15] and Barndorff-Nielsen, Blæsild and Schmiegel[2], respectively. Time change in a broad mathematical sense is treated in Barndorff-Nielsen and Shiryaev[8].

2. Some notation and terminology

Lévy processes, additive processes, and H-selfsimilar (i. e. selfsimilar with exponent $H > 0$) processes in this paper are in the sense of Sato[34]. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the sets of positive integers, integers, rational numbers, real numbers, and complex numbers, respectively; $\mathbb{R}^d$ is the d-dimensional Euclidean space; elements of $\mathbb{R}^d$ are column d-vectors; the canonical inner product and norm are denoted by $\langle x, y\rangle$ and $|x|$ for $x, y \in \mathbb{R}^d$; $\mathcal{B}(\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}^d$. A cone K in $\mathbb{R}^d$ is a non-empty closed convex set which is closed under multiplication by nonnegative reals, contains no straight line through 0, and such that $K \neq \{0\}$. $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^d_+ = [0,\infty)^d$, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, and $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$.

The distribution of an $\mathbb{R}^d$-valued random variable X is denoted by $\mathcal{L}(X)$. Furthermore, $\widehat{\mu}(z)$ is the characteristic function of a distribution μ and $C_\mu(z)$ is the cumulant function of μ for which $\widehat{\mu}(z) \neq 0$ for all z, that is, the continuous function with $C_\mu(0) = 0$ such that $\widehat{\mu}(z) = \exp(C_\mu(z))$. When $\mu = \mathcal{L}(X)$, we also write this as $C_X(z)$. The support of μ is denoted by $\mathrm{Supp}(\mu)$.

For two random variables X and Y, $X \overset{d}{=} Y$ means that X and Y have a common distribution. For two stochastic processes $X = \{X_t\}$ and $Y = \{Y_t\}$, $X \overset{d}{=} Y$ or $\{X_t\} \overset{d}{=} \{Y_t\}$ means that X and Y have a common system of finite-dimensional marginals.

Let $0 < \alpha \leqslant 2$. A distribution μ on $\mathbb{R}^d$ is called strictly α-stable if μ is infinitely divisible and, for any $c \in (0,\infty)$, $\widehat{\mu}(z)^c = \widehat{\mu}(c^{1/\alpha} z)$; μ is called α-stable if μ is infinitely divisible and, for any $c \in (0,\infty)$, there is $\gamma_c \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^c = \widehat{\mu}(c^{1/\alpha} z) e^{i\langle \gamma_c, z\rangle}$.

Denote by $L_0(\mathbb{R}^d)$ the class of all selfdecomposable distributions on $\mathbb{R}^d$. That is, $\mu \in L_0(\mathbb{R}^d)$ if and only if μ is a distribution on $\mathbb{R}^d$ such that, for any $c \in (0,1)$, there is a distribution $\rho^{(c)}$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(cz)\widehat{\rho^{(c)}}(z), \qquad z \in \mathbb{R}^d. \tag{2.1}$$

If μ is selfdecomposable, then μ is infinitely divisible and, for each c, $\rho^{(c)}$ is unique and infinitely divisible.

Let $m \in \mathbb{N}$. Denote by $L_m(\mathbb{R}^d)$ the class of $\mu \in L_0(\mathbb{R}^d)$ such that, for any $c \in (0,1)$, (2.1) holds with some $\rho^{(c)} \in L_{m-1}(\mathbb{R}^d)$. Denote $L_\infty(\mathbb{R}^d) = \bigcap_{0\leqslant m<\infty} L_m(\mathbb{R}^d)$. For $m \in \mathbb{N} \cup \{\infty\}$, a distribution $\mu \in L_m(\mathbb{R}^d)$ is said to be of class L_m. As in Maejima and Sato[25] and Sato[35], a process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is called a semi-Lévy process if it is an additive process and if there is $p > 0$ such that $X_t - X_s \overset{d}{=} X_{t+p} - X_{s+p}$ for all $0 \leqslant s \leqslant t$.

Let $\mathfrak{T}$ be the family of all non-empty finite subsets of $\mathbb{R}_+$. Denote by $\#\tau$ the cardinality of τ. For a stochastic process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ and $\tau = \{t_1, ..., t_n\} \in \mathfrak{T}$ with $\#\tau = n$, denote $X_\tau = \{X_t : t \in \tau\} = \{X_{t_j} : j = 1, \ldots, n\}$. For $K \subset \mathbb{R}^d$ and $\tau \in \mathfrak{T}$, denote $K^\tau = \{x = (x_t)_{t\in\tau} : x_t \in K \text{ for all } t \in \tau\}$. Similarly, $K^{\mathbb{R}_+} = \{x = (x_t)_{t\in\mathbb{R}_+} : x_t \in K \text{ for all } t \in \mathbb{R}_+\}$. For $a > 0$ and $\tau = \{t_1, ..., t_n\} \in \mathfrak{T}$ with $\#\tau = n$, we use the notation $a\tau = \{at_1, ..., at_n\}$.

3. Infinite divisibility and selfdecomposability of processes

We discuss infinite divisibility, selfdecomposability, class L_m property, and stability of stochastic processes and their weak versions.

Definition 3.1. A stochastic process $X = \{X_t\colon t \geqslant 0\}$ on $\mathbb{R}^d$ is *infinitely divisible* (resp. *selfdecomposable*; resp. *of class* L_m) if all finite-dimensional marginals of X are infinitely divisible (resp. selfdecomposable; resp. of class L_m), that is, for any choice of $\tau \in \mathfrak{T}$, $\mathcal{L}(X_\tau)$ is an infinitely divisible (resp. selfdecomposable; resp. of class L_m) distribution on the $((\#\tau)d)$-dimensional Euclidean space $\mathbb{R}^{(\#\tau)d}$.

Obviously a Lévy process or, more generally, an additive process on $\mathbb{R}^d$ is an infinitely divisible process.

Definition 3.2. A stochastic process $X = \{X_t\colon t \geqslant 0\}$ on $\mathbb{R}^d$ is *weakly infinitely divisible* (resp. *weakly selfdecomposable*; resp. *weakly of class* L_m) if, for any choice of $\tau = \{t_1, \ldots, t_n\} \in \mathfrak{T}$ and for any $a_1, \ldots, a_n \in \mathbb{R}$, $\sum_{j=1}^n a_j X_{t_j}$ is infinitely divisible (resp. selfdecomposable; resp. of class L_m).

These "weak" concepts are strictly weaker than the original concepts. See Proposition 3.12.

Remark 3.3. Let us call a function $f(t)$ on $[0,\infty)$ a *step function* if, for some $0 = t_0 < t_1 < \cdots < t_n < \infty$ and $a_1, \ldots, a_n \in \mathbb{R}$, $f(t) = \sum_{j=1}^n a_j 1_{(t_{j-1},t_j]}(t)$. Let $X = \{X_t\colon t \geqslant 0\}$ be a stochastic process on $\mathbb{R}^d$. For any step function of the form above we write

$$f \cdot X = \int_0^\infty f(t) dX_t = \sum_{j=1}^n a_j (X_{t_j} - X_{t_{j-1}}).$$

Assume that $X_0 = 0$ a.s. Then X is weakly infinitely divisible (resp. weakly selfdecomposable; resp. weakly of class L_m) if and only if, for any step function f, $f \cdot X$ is infinitely divisible (resp. selfdecomposable; resp. of class L_m).

We proceed to develop generalizations to $\mathbb{R}^d$ of some of the results of Maruyama[28].

Let $k \in \mathbb{N}$ (usually $k = nd$ with $n \in \mathbb{N}$) and let $a(x) = (x \vee (-1)) \wedge 1$ for $x \in \mathbb{R}$. For any infinitely divisible distribution μ on $\mathbb{R}^k$, we sometimes use the Lévy–Khintchine representation of the form

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^k} \Big(e^{i\langle z,x\rangle} - 1 - i \sum_{j=1}^k a(x_j) z_j \Big) \nu(dx) + i\langle \gamma, z\rangle \tag{3.1}$$

for $z = (z_j)_{1\leqslant j\leqslant k} \in \mathbb{R}^k$, where $x = (x_j)_{1\leqslant j\leqslant k}$, A is a $k \times k$ symmetric nonnegative-definite matrix, ν is a measure (Lévy measure) on $\mathbb{R}^k$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^k} (1 \wedge |x|^2)\nu(dx) < \infty$, and $\gamma \in \mathbb{R}^k$. The triplet of A, ν, and γ is denoted by $(A, \nu, \gamma)_a$.

Let K be a cone in $\mathbb{R}^k$. A distribution μ is infinitely divisible and satisfies $\mathrm{Supp}(\mu) \subset K$ if and only if

$$C_\mu(z) = \int_K (e^{i\langle z,x\rangle} - 1)\nu(dx) + i\langle \gamma^0, z\rangle \tag{3.2}$$

for $z \in \mathbb{R}^k$, where ν is a measure (Lévy measure) on $\mathbb{R}^k$ satisfying $\mathrm{Supp}(\nu) \subset K$, $\nu(\{0\}) = 0$, and $\int_K (1 \wedge |x|)\nu(dx) < \infty$, and $\gamma^0 \in K$; γ^0 is called the drift (see Skorohod[36] or Sato[34] E 22.11). In this case we say that μ has triplet $(0, \nu, \gamma^0)_0$. It follows from (3.2) that

$$\int_{\mathbb{R}^k} e^{-\langle z,x\rangle}\mu(dx) = \exp\left[\int_K (e^{-\langle z,x\rangle} - 1)\nu(dx) - \langle \gamma^0, z\rangle\right] \tag{3.3}$$

for $z \in \mathbb{C}^k$ satisfying $\mathrm{Re}\,\langle z, x\rangle \geqslant 0$ for all $x \in K$ (see Sato[34] Theorem 25.17 or Pedersen and Sato[29]); this fact will be used in Section 7. Here we are using $\langle z, x\rangle = \langle x, z\rangle = \sum_{j=1}^k z_j x_j$ even for $z = (z_j)_{1\leqslant j\leqslant k} \in \mathbb{C}^k$.

For $\tau, \tau' \in \mathfrak{T}$ satisfying $\tau \subset \tau'$, let $f_{\tau\tau'}$ be the ordinary projection from $(\mathbb{R}^d)^{\tau'}$ onto $(\mathbb{R}^d)^\tau$; $f_{\tau\tau}$ is the identity map from $(\mathbb{R}^d)^\tau$ onto itself. For $\tau \in \mathfrak{T}$, let $f_{\tau\mathbb{R}_+}$ be the ordinary projection from $(\mathbb{R}^d)^{\mathbb{R}_+}$ onto $(\mathbb{R}^d)^\tau$. Thus, $X_\tau = f_{\tau\mathbb{R}_+}X$ for any stochastic process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ and $\tau \in \mathfrak{T}$.

Theorem 3.4. *Let $X = \{X_t \colon t \geqslant 0\}$ be an infinitely divisible process on $\mathbb{R}^d$. Then there are $\overline{\nu} = \{\nu_\tau \colon \tau \in \mathfrak{T}\}$, $A = (A_{t,p,u,q})_{t,u\in\mathbb{R}_+,\,p,q=1,\dots,d}$, and $\gamma \in (\mathbb{R}^d)^{\mathbb{R}_+}$ such that the following are satisfied for all $\tau \in \mathfrak{T}$:*

(a) *ν_τ is a measure on $(\mathbb{R}^d)^\tau$ with $\nu(\{0\}) = 0$ and $\int_{(\mathbb{R}^d)^\tau} (1 \wedge |x|^2)\nu_\tau(dx) < \infty$;*

(b) *if $B \in \mathcal{B}((\mathbb{R}^d)^\tau)$ and $0 \notin B$, then $\nu_\tau(B) = \nu_{\tau'}(f_{\tau\tau'}^{-1}(B))$ for any $\tau' \in \mathfrak{T}$ with $\tau \subset \tau'$;*

(c) *the restriction $A_\tau = (A_{t,p,u,q})_{t,u\in\tau,\,p,q=1,\dots,d}$ of A to τ is a symmetric, nonnegative-definite $(\#\tau)d \times (\#\tau)d$ matrix;*

(d) *X_τ has the triplet $(A_\tau, \nu_\tau, \gamma_\tau)_a$, where $\gamma_\tau = f_{\tau\mathbb{R}_+}\gamma$.*

Conversely, for any $\overline{\nu}, A, \gamma$ satisfying (a), (b), *and* (c) *for all $\tau \in \mathfrak{T}$, there exists an infinitely divisible process X on $\mathbb{R}^d$ satisfying* (d) *for all $\tau \in \mathfrak{T}$.*

Theorem 3.4 is a reformulation and $\mathbb{R}^d$-generalization of Theorem 1 (and the remark following it) of Maruyama[28], who treated only the case $d = 1$ and constructed a 'big' Lévy measure on $\mathbb{R}^{\mathbb{R}_+}$.

Outline of the proof of Theorem 3.4. Let X be an infinitely divisible process on $\mathbb{R}^d$ and let $(A_\tau, \nu_\tau, \gamma_\tau)_a$ be the triplet of X_τ based on a. We represent $x \in (\mathbb{R}^d)^\tau$ as $x = (x_t)_{t\in\tau} = ((x_{t,j})_{1\leqslant j\leqslant d})_{t\in\tau}$. Let $\tau \subset \tau'$. Then, for $z \in (\mathbb{R}^d)^\tau$,

$$C_{X_\tau}(z) = C_{f_{\tau\tau'}X_{\tau'}}(z) = C_{X_{\tau'}}(z'),$$

where z' is defined by z as

$$z' \in (\mathbb{R}^d)^{\tau'}, \qquad f_{\tau\tau'}z' = z, \qquad f_{(\tau'\setminus\tau),\tau'}z' = 0. \tag{3.4}$$

We have

$$C_{X_{\tau'}}(z') = -\tfrac{1}{2}\langle z', A_{\tau'}z'\rangle + I + i\langle \gamma_{\tau'}, z'\rangle,$$

where the second term in the right-hand side is as follows:

$$\begin{aligned} I &= \int_{(\mathbb{R}^d)^{\tau'}} \left[e^{i\langle z',x\rangle} - 1 - i\sum_{t\in\tau'}\sum_{j=1}^{d} a(x_{t,j})z'_{t,j} \right] \nu_{\tau'}(dx) \\ &= \int_{(\mathbb{R}^d)^{\tau'}} \left[e^{i\langle z, f_{\tau\tau'}x\rangle} - 1 - i\sum_{t\in\tau}\sum_{j=1}^{d} a(x_{t,j})z_{t,j} \right] \nu_{\tau'}(dx) \\ &= \int_{(\mathbb{R}^d)^{\tau}} \left[e^{i\langle z,x\rangle} - 1 - i\sum_{t\in\tau}\sum_{j=1}^{d} a(x_{t,j})z_{t,j} \right] (\nu_{\tau'}f_{\tau\tau'}^{-1})(dx). \end{aligned}$$

Hence (a), (b), and (c) are satisfied and $\gamma_\tau = f_{\tau\tau'}\gamma_{\tau'}$. (The frequently used Lévy–Khintchine representations, different from (3.1), do not allow such a manipulation as above.) The converse part of the theorem is proved by applying Kolmogorov's extension theorem.

Similarly we can prove the following.

Theorem 3.5. *Let K be a cone in $\mathbb{R}^d$. Let $X = \{X_t : t \geqslant 0\}$ be an infinitely divisible process on $\mathbb{R}^d$. Assume that*

$$P(X_\tau \in K^\tau) = 1 \qquad \text{for } \tau \in \mathfrak{T}. \tag{3.5}$$

Then there are $\overline{\nu} = \{\nu_\tau : \tau \in \mathfrak{T}\}$ and $\gamma^0 \in K^{\mathbb{R}_+}$ such that the following are satisfied for all $\tau \in \mathfrak{T}$:

(a) *ν_τ is a measure on K^τ with $\nu(\{0\}) = 0$ and $\int_{K^\tau}(1 \wedge |x|)\nu_\tau(dx) < \infty$;*

(b) *if $B \in \mathcal{B}((\mathbb{R}^d)^\tau)$ and $0 \notin B$, then $\nu_\tau(B) = \nu_{\tau'}(f_{\tau\tau'}^{-1}(B))$ for any $\tau' \in \mathfrak{T}$ with $\tau \subset \tau'$;*

(c) *X_τ has the triplet $(0, \nu_\tau, \gamma_\tau^0)_0$, where $\gamma_\tau^0 = f_{\tau\mathbb{R}_+}\gamma^0$.*

Conversely, for any $\overline{\nu}$ and γ^0 satisfying (a) *and* (b) *for all $\tau \in \mathfrak{T}$, there exists an infinitely divisible process X on $\mathbb{R}^d$ satisfying* (3.5) *and* (c) *for all $\tau \in \mathfrak{T}$.*

Let us give equivalent conditions in terms of stochastic processes for infinitely divisibility, selfdecomposability, and the L_m property of stochastic processes.

Theorem 3.6. *A stochastic process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is infinitely divisible if and only if, for each $k \in \mathbb{N}$, there are independent, identically distributed stochastic processes $X^{(k,1)}, \dots, X^{(k,k)}$ on $\mathbb{R}^d$ such that*

$$X \stackrel{d}{=} X^{(k,1)} + \cdots + X^{(k,k)}. \tag{3.6}$$

If X is infinitely divisible, then the law of $X^{(k,1)}$ is uniquely determined by the law of X and k, and the process $X^{(k,1)}$ is infinitely divisible. If X is furthermore stochastically continuous, then $X^{(k,1)}$ is also stochastically continuous.

Proof. The "if" part. For any $\tau \in \mathfrak{T}$ and k,

$$\mathcal{L}(X_\tau) = \mathcal{L}(X_\tau^{(k,1)} + \cdots + X_\tau^{(k,k)}) = \mathcal{L}(X_\tau^{(k,1)})^{k*}.$$

Hence $\mathcal{L}(X_\tau)$ is infinitely divisible.

The "only if" part. The infinitely divisible process X induces A, $\overline{\nu} = \{\nu_\tau \colon \tau \in \mathfrak{T}\}$, and γ as in Theorem 3.4. Let

$$A^{(k)} = k^{-1}A, \qquad \nu_\tau^{(k)} = k^{-1}\nu_\tau, \qquad \gamma^{(k)} = k^{-1}\gamma.$$

Then $A^{(k)}$, $\overline{\nu}^{(k)} = \{\nu_\tau^{(k)} \colon \tau \in \mathfrak{T}\}$, and $\gamma^{(k)}$ satisfy (a), (b), and (c) for all $\tau \in \mathfrak{T}$. Hence there is an infinitely divisible process $X^{(k)}$ such that, for any $\tau \in \mathfrak{T}$, $X_\tau^{(k)}$ has triplet $(A_\tau^{(k)}, \nu_\tau^{(k)}, \gamma_\tau^{(k)})_a$. Let $X^{(k,1)}, \ldots, X^{(k,k)}$ be independent copies of $X^{(k)}$. Then we have (3.6).

Uniqueness. Since any infinitely divisible distribution has a unique kth convolution root, $\mathcal{L}(X_\tau^{(k)})$ is uniquely determined by $\mathcal{L}(X_\tau)$ and k. It is infinitely divisible.

Stochastic continuity. X is stochastically continuous if and only if, for any $t \geqslant 0$, $\mathcal{L}(X_s - X_t) \to \delta_0$ as $s \to t$, that is, $Ee^{i\langle z, X_s - X_t\rangle} \to 1$, $z \in \mathbb{R}^d$, as $s \to t$. If an infinitely divisible process X is stochastically continuous, then $X^{(k,1)}$ is stochastically continuous for each k, because

$$\begin{aligned} E\exp(i\langle z, X_s - X_t\rangle) &= E\exp\left(i\sum_{l=1}^{k}\langle z, X_s^{(k,l)} - X_t^{(k,l)}\rangle\right) \\ &= \left(E\exp(i\langle z, X_s^{(k,1)} - X_t^{(k,1)}\rangle)\right)^k. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.7. *A stochastic process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ is selfdecomposable if and only if, for every $c \in (0,1)$,*

$$X \stackrel{d}{=} cX' + U^{(c)}, \tag{3.7}$$

where $X' = \{X'_t \colon t \geqslant 0\}$ is a copy of X, $U^{(c)} = \{U_t^{(c)} \colon t \geqslant 0\}$ is a stochastic process on $\mathbb{R}^d$, and X' and $U^{(c)}$ are independent. The law of $U^{(c)}$ is uniquely determined by c and the law of X. The process $U^{(c)}$ is infinitely divisible.

Proof. Obviously, the existence of independent X' and $U^{(c)}$ satisfying (3.7) implies that X is selfdecomposable. Conversely, suppose that X is selfdecomposable. Let $c \in (0,1)$. For every $\tau \in \mathfrak{T}$, denote $\mu_\tau = \mathcal{L}(X_\tau)$. We have

$$\widehat{\mu}_\tau(z) = \widehat{\mu}_\tau(cz)\widehat{\rho_\tau^{(c)}}(z), \qquad z \in (\mathbb{R}^d)^\tau, \tag{3.8}$$

with some distribution $\rho_\tau^{(c)}$ on $(\mathbb{R}^d)^\tau$. To show the consistency of the system $\{\rho_\tau^{(c)} : \tau \in \mathfrak{T}\}$, let $\tau, \tau' \in \mathfrak{T}$ with $\tau \subset \tau'$. We claim that

$$\rho_\tau^{(c)}(B) = \rho_{\tau'}^{(c)}(f_{\tau\tau'}^{-1}(B)), \qquad B \in \mathcal{B}((\mathbb{R}^d)^\tau). \tag{3.9}$$

This is equivalent to

$$\widehat{\rho_\tau^{(c)}}(z) = \widehat{\rho_{\tau'}^{(c)}}(z'), \tag{3.10}$$

where z and z' are related by (3.4). Compare (3.8) with $\widehat{\mu}_{\tau'}(z') = \widehat{\mu_{\tau'}}(cz')\widehat{\rho}_{\tau'}^{(c)}(z')$, $z' \in (\mathbb{R}^d)^{\tau'}$ and note that (3.4) implies $\widehat{\mu}_\tau(z) = \widehat{\mu}_{\tau'}(z')$ and $\widehat{\mu}_\tau(cz) = \widehat{\mu}_{\tau'}(cz')$. Then we have $\widehat{\rho_\tau^{(c)}}(z) = \widehat{\rho_{\tau'}^{(c)}}(z')$. Therefore we get (3.10) and consequently (3.9). By Kolmogorov's extension theorem there is a stochastic process $U^{(c)}$ such that $\mathcal{L}(U_\tau^{(c)}) = \rho_\tau^{(c)}$. Construct X' so that X' and $U^{(c)}$ are independent and $X' \overset{d}{=} X$. Then it follows from (3.8) that $X \overset{d}{=} cX' + U^{(c)}$. Since μ_τ is infinitely divisible, the value of $\widehat{\mu}_\tau$ is non-zero. Thus $\rho_\tau^{(c)}$ is uniquely determined by (3.8). See Sato[34] Proposition 15.5 for the infinite divisibility of $\rho_\tau^{(c)}$. $\square$

Theorem 3.8. *Let X be a selfdecomposable process on $\mathbb{R}^d$. If X is an additive (resp. Lévy) process in law, then, for every $c \in (0,1)$, the process $U^{(c)}$ in Theorem 3.7 is an additive (resp. Lévy) process in law.*

Here an additive (or Lévy) process in law is in the sense of Sato[34]. Any additive (resp. Lévy) process in law on $\mathbb{R}^d$ has an additive (resp. Lévy) process modification.

Proof of theorem. We have

$$X_\tau \overset{d}{=} cX'_\tau + U_\tau^{(c)} \tag{3.11}$$

for each $\tau \in \mathfrak{T}$, where X' and $U^{(c)}$ are independent and $X' \overset{d}{=} X$. As $s \uparrow t$ or $t \downarrow s$, $E\exp(i\langle z, X_t - X_s\rangle) \to 1$. Since

$$E\exp(i\langle z, X_t - X_s\rangle) = E\exp(i\langle z, cX'_t - cX'_s\rangle)E\exp(i\langle z, U_t^{(c)} - U_s^{(c)}\rangle),$$

it follows that $E\exp(i\langle z, U_t^{(c)} - U_s^{(c)}\rangle) \to 1$. Hence $U^{(c)}$ is stochastically continuous. We have $U_0^{(c)} = 0$ a.s. since $X_0 = 0$ a.s. Let $0 = t_0 < t_1 < \cdots < t_n$. Use (3.11) for $\tau = \{t_0, \ldots, t_n\}$. Then, for $z_1, \ldots, z_n \in \mathbb{R}^d$,

$$\begin{aligned}
&E\exp\left(i\sum_{j=1}^n \langle z_j, U_{t_j}^{(c)} - U_{t_{j-1}}^{(c)}\rangle\right) \\
&\quad = E\exp\left(i\sum_{j=1}^n \langle z_j, X_{t_j} - X_{t_{j-1}}\rangle\right) \Bigg/ E\exp\left(i\sum_{j=1}^n \langle z_j, cX'_{t_j} - cX'_{t_{j-1}}\rangle\right) \\
&\quad = \prod_{j=1}^n E\exp(i\langle z_j, X_{t_j} - X_{t_{j-1}}\rangle) \Bigg/ \prod_{j=1}^n E\exp(i\langle z_j, cX'_{t_j} - cX'_{t_{j-1}}\rangle)
\end{aligned}$$

$$= \prod_{j=1}^{n} E\exp(i\langle z_j, U^{(c)}_{t_j} - U^{(c)}_{t_{j-1}}\rangle).$$

Hence $U^{(c)}$ as independent increments. This shows that $U^{(c)}$ is an additive process in law. If X is a Lévy process in law, then

$$\begin{aligned} E\exp(i\langle z, U^{(c)}_t - U^{(c)}_s\rangle) &= E\exp(i\langle z, X_t - X_s\rangle) \,/ E\exp(i\langle z, cX'_t - cX'_s\rangle) \\ &= E\exp(i\langle z, X_{t-s}\rangle) \,/ E\exp(i\langle z, cX'_{t-s}\rangle) = E\exp(i\langle z, U^{(c)}_{t-s}\rangle) \end{aligned}$$

and hence $U^{(c)}$ is a Lévy process in law. $\square$

Theorem 3.9. *Let X be a selfdecomposable process on $\mathbb{R}^d$ and let $H > 0$. Then X is H-selfsimilar if and only if, for every $c \in (0,1)$, the process $U^{(c)}$ in Theorem 3.7 is H-selfsimilar.*

Proof. Suppose that X is H-selfsimilar. Then $X_{a\tau} \stackrel{d}{=} a^H X_\tau$ for any $a > 0$ and $\tau \in \mathfrak{T}$. It follows from (3.11) that

$$\begin{aligned} E\exp(i\langle z, U^{(c)}_{a\tau}\rangle) &= E\exp(i\langle z, X_{a\tau}\rangle) \,/ E\exp(i\langle z, cX'_{a\tau}\rangle) \\ &= E\exp(i\langle z, a^H X_\tau\rangle) \,/ E\exp(i\langle z, ca^H X'_\tau\rangle) = E\exp(i\langle z, a^H U^{(c)}_\tau\rangle). \end{aligned}$$

Hence $U^{(c)}$ is H-selfsimilar. Conversely, if $U^{(c)}$ is H-selfsimilar for every $c \in (0,1)$, then, letting $c \downarrow 0$ in $E\exp(i\langle z, U^{(c)}_{a\tau}\rangle) = E\exp(i\langle z, a^H U^{(c)}_\tau\rangle)$, we get $E\exp(i\langle z, X_{a\tau}\rangle) = E\exp(i\langle z, a^H X_\tau\rangle)$. Note that, for any $\tau \in \mathfrak{T}$,

$$E\exp(i\langle z, U^{(c)}_\tau\rangle) = E\exp(i\langle z, X_\tau\rangle) \,/ E\exp(i\langle z, cX'_\tau\rangle) \to E\exp(i\langle z, X_\tau\rangle)$$

as $c \downarrow 0$. $\square$

Example 3.10. (An application of Theorem 3.7.) Let $X = \{X_t\}$ be a selfdecomposable and H-selfsimilar additive process on $\mathbb{R}$. Let $a \in (0,1) \cup (1,\infty)$. Define

$$Y_t = X_t + X_{at}.$$

Then $Y = \{Y_t\}$ is a selfdecomposable and H-selfsimilar process, but Y is not an additive process in general. Indeed, let X' and $U^{(c)}$ be the processes in Theorem 3.7. Then

$$\{X_t + X_{at}\} \stackrel{d}{=} \{cX'_t + cX'_{at} + U^{(c)}_t + U^{(c)}_{at}\},$$

and we see that Y is selfdecomposable. For any $b > 0$, $\{X_{bt}\} \stackrel{d}{=} \{b^H X_t\}$. Hence $\{X_{bt} + X_{abt}\} \stackrel{d}{=} \{b^H X_t + b^H X_{at}\}$, that is, Y is H-selfsimilar. Now assume that $E|X_t|^2 < \infty$, $EX_t = 0$, and $E(X_t - X_s)^2 \neq 0$ for $0 \leqslant s < t$. Then $E|Y_t|^2 < \infty$ and $EY_t = 0$. In order to show that Y is not an additive process, it suffices to check $E(Y_t - Y_s)Y_s \neq 0$ for $0 < s < t$. We have

$$(Y_t - Y_s)Y_s = (X_t - X_s)X_s + (X_{at} - X_{as})X_{as} + (X_t - X_s)X_{as} + (X_{at} - X_{as})X_s.$$

If $a > 1$, then the third term has nonzero mean but the other terms have zero mean. If $0 < a < 1$, then the fourth term has nonzero mean but the other terms have zero mean.

Theorem 3.11. *Let $m \in \mathbb{N} \cup \{\infty\}$. A stochastic process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is of class L_m if and only if, for every $c \in (0,1)$,*

$$X \stackrel{d}{=} cX' + U^{(c)}, \tag{3.12}$$

where X' is a copy of X, $U^{(c)}$ is a process of class L_{m-1}, and X' and $U^{(c)}$ are independent. Here we understand $m - 1 = \infty$ for $m = \infty$.

Proof. Assume that X is of class L_m. Then, by Theorem 3.7, (3.12) is true with some infinitely divisible process $U^{(c)}$. For any $\tau \in \mathfrak{T}$, $X_\tau \stackrel{d}{=} cX'_\tau + U^{(c)}_\tau$ and X'_τ and $U^{(c)}_\tau$ are independent. It follows that $\mathcal{L}(U^{(c)}_\tau) \in L_{m-1}$. Thus $U^{(c)}$ is of class L_{m-1}. The converse is proved similarly. $\square$

Proposition 3.12. *Let $m \in \mathbb{N}$. If a stochastic process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is infinitely divisible (resp. selfdecomposable; resp. of class L_m), then it is weakly infinitely divisible (resp. weakly selfdecomposable; resp. weakly of class L_m). But the converse is not true.*

Proof. The first assertion follows from the fact that if $\mathcal{L}((X_{t_j})_{1 \leqslant j \leqslant n})$ is infinitely divisible (resp. selfdecomposable; resp. of class L_m), then $\mathcal{L}(\sum_{j=1}^n a_j X_{t_j})$ is infinitely divisible (resp. selfdecomposable; resp. of class L_m). This is a special case of Sato(34) Proposition 11.10 (resp. Maejima, Sato and Watanabe(26) Lemma 1). The last assertion follows from Sato(34) E 12.4 (resp. Sato(33) Theorem 1.1, or Maejima, Suzuki and Tamura(27) Theorem 1). Indeed, let $d = 1$, $n \geqslant 2$, and let $Z_1, \ldots, Z_n$ be such that $\mathcal{L}((Z_j)_{1 \leqslant j \leqslant n})$ is not infinitely divisible (resp. not selfdecomposable; resp. not of class L_m) but that all linear combinations of $Z_1, \ldots, Z_n$ are infinitely divisible (resp. selfdecomposable; resp. of class L_m). Define $X = \{X_t : t \geqslant 0\}$ as $X_0 = 0$, $X_j = Z_j$ for $j = 1, \ldots, n$,

$$X_t = (j + 1 - t)Z_j + (t - j)Z_{j+1} \qquad \text{for } j \leqslant t \leqslant j + 1 \qquad (j = 1, \ldots, n),$$

and $X_t = Z_n$ for $t \geqslant n$. Then X is weakly infinitely divisible (resp. weakly selfdecomposable; resp. weakly of class L_m) but not infinitely divisible (resp. not selfdecomposable; resp. not of class L_m). $\square$

In spite of the proposition above, Lévy processes or, more generally, additive processes have the following property.

Theorem 3.13. *Suppose that $X = \{X_t : t \geqslant 0\}$ is an additive process on $\mathbb{R}^d$. Let $m \in \mathbb{N} \cup \{0, \infty\}$. Then the following are equivalent:*

(a) *X is of class L_m;*

(b) *X is weakly of class L_m;*

(c) *$\mathcal{L}(X_t - X_s) \in L_m(\mathbb{R}^d)$ for all $0 \leqslant s < t$.*

Proof. See Theorem 1 of Maejima, Sato and Watanabe[26]. It treats selfsimilar additive processes on $\mathbb{R}^d$. But the proof of the equivalence of (a), (b), and (c) does not use the selfsimilarity. $\square$

Remark 3.14. If X is a selfsimilar additive process on $\mathbb{R}^d$, then (a), (b), (c) are equivalent to

(d) $\mathcal{L}(X_1) \in L_{m+1}(\mathbb{R}^d)$.

Here we understand $m+1 = \infty$ if $m = \infty$. This is Theorem 1 of Maejima, Sato and Watanabe[26]. If X is a Lévy process on $\mathbb{R}^d$, then, obviously, (c) is equivalent to $\mathcal{L}(X_1) \in L_m(\mathbb{R}^d)$.

Definition 3.15. A stochastic process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ is *α-stable* (resp. *strictly α-stable*) with $0 < \alpha \leqslant 2$ if all its finite-dimensional marginals are α-stable (resp. strictly α-stable).

Definition 3.16. A stochastic process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ is *weakly α-stable* (resp. *weakly strictly α-stable*) if all finite linear combinations of X_t, $t \geqslant 0$, are α-stable (resp. strictly α-stable).

Remark 3.17. A stochastic process X on $\mathbb{R}$ (that is, $d = 1$) is α-stable with $1 \leqslant \alpha \leqslant 2$ (resp. strictly α-stable with $0 < \alpha \leqslant 2$) if and only if X is weakly α-stable with $1 \leqslant \alpha \leqslant 2$ (resp. weakly strictly α-stable with $0 < \alpha \leqslant 2$). See Samorodnitsky and Taqqu[31] Theorem 2.1.5. We do not know whether this is true in the case $d \geqslant 2$; it *seems* to us that the "if" part is not true even when $\alpha = 2$. If X is an α-stable process on $\mathbb{R}$ with $0 < \alpha < 1$, then X is weakly α-stable with the same α. But the converse is not true; see Samorodnitsky and Taqqu[31] for references.

Definition 3.18. A stochastic process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ is said to have *finite log-moment* if $E \log^+ |X_t| < \infty$ for all t, where $\log^+ u = (\log u) \vee 0$. (If $\mathcal{L}(X_t)$ is infinitely divisible, then this condition is equivalent to saying that $\int \log^+ |x| \nu_t(dx) < \infty$ for all t, where ν_t is the Lévy measure of $\mathcal{L}(X_t)$.)

4. Stationary processes of Ornstein-Uhlenbeck type

In this section, we consider stationary processes of Ornstein-Uhlenbeck type on $\mathbb{R}^d$ (in short, stationary OU process), i.e. the stationary solution of a stochastic differential equation of the form

$$dV_t = -\lambda V_t dt + dZ_{\lambda t} \tag{4.1}$$

where Z, called the background driving Lévy process (BDLP) has finite log-moment and $\lambda > 0$. For all such processes V, the marginal law $\mathcal{L}(V_t)$ for each $t \geqslant 0$ is selfdecomposable and does not depend on λ.

Theorem 4.1. *Let $V = \{V_t\colon t \geqslant 0\}$ be a stationary OU process on $\mathbb{R}^d$. Then V is an infinitely divisible process.*

Proof. To say that V is a stationary OU process satisfying (4.1) is equivalent to saying that

$$V_t = e^{-\lambda t} V_0 + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}, \qquad t \geqslant 0 \tag{4.2}$$

with additional conditions that V_0 and Z are independent and that $V_0 \stackrel{d}{=} \int_0^\infty e^{-\lambda s} dZ_{\lambda s}$. Recall that $\int_0^\infty e^{-\lambda s} dZ_{\lambda s}$ exists if and only if Z has finite log-moment. For any $k \in \mathbb{N}$ there exist independent identically distributed Lévy processes $\{Z_{\lambda s}^{(k,l)}\}$, $l = 1, 2, ..., k$, with finite log-moment such that $Z \stackrel{d}{=} Z^{(k,1)} + \cdots + Z^{(k,k)}$. It follows that, for any $\tau = \{t_1, ..., t_n\} \in \mathfrak{T}$,

$$V_\tau \stackrel{d}{=} \sum_{l=1}^k \left(e^{-\lambda t_j} V_0^{(k.l)} + \int_0^{t_j} e^{-\lambda(t_j - s)} dZ_{\lambda s}^{(k,l)} \right)_{1 \leqslant j \leqslant n}$$

where $V_0^{(k,1)}, \ \ldots, \ V_0^{(k,k)}, \ \{Z_{\lambda s}^{(k,1)}\}, \ \ldots, \ \{Z_{\lambda s}^{(k,k)}\}$ are independent and $V_0^{(k,l)} \stackrel{d}{=} \int_0^\infty e^{-\lambda s} dZ_{\lambda s}^{(k,l)}$. Thus V_τ is infinitely divisible for any $\tau \in \mathfrak{T}$. $\square$

Theorem 4.2. *Let V be a stationary OU process on $\mathbb{R}^d$ with the BDLP Z. Then the following three conditions are equivalent:*

(a) *V is a selfdecomposable process;*

(b) *Z is a selfdecomposable process;*

(c) *$\mathcal{L}(V_t)$ is of class L_1 for each $t \geqslant 0$.*

Proof. Let us prove that (b) implies (a). By Theorem 3.8 we see that for each $c \in (0, 1)$ there exists a Lévy process $U^{(c)}$ such that $Z \stackrel{d}{=} cZ' + U^{(c)}$, where Z' is a copy of Z and Z' and $U^{(c)}$ are independent. It follows from (4.2) that

$$V \stackrel{d}{=} \left\{ e^{-\lambda t} V_0 + c \int_0^t e^{-\lambda(t-s)} dZ'_{\lambda s} + \int_0^t e^{-\lambda(t-s)} dU_{\lambda s}^{(c)} : t \geqslant 0 \right\}$$

where V_0, Z', and $U^{(c)}$ are independent and $V_0 \stackrel{d}{=} \int_0^\infty e^{-\lambda s} dZ_{\lambda s}$. Repeating the argument in Sato[34] p. 161, we see that $U^{(c)}$ has finite log-moment. It follows that $\int_0^\infty e^{-\lambda s} dU_{\lambda s}^{(c)}$ exists. Hence

$$\begin{aligned} V \stackrel{d}{=} \Bigg\{ & c\, e^{-\lambda t} \int_0^\infty e^{-\lambda s} dZ''_{\lambda s} + e^{-\lambda t} \int_0^\infty e^{-\lambda s} dU_{\lambda s}^{(c)\prime} \\ & + c \int_0^t e^{-\lambda(t-s)} dZ'_{\lambda s} + \int_0^t e^{-\lambda(t-s)} dU_{\lambda s}^{(c)} \Bigg\} \end{aligned}$$

where Z'', $U^{(c)\prime}$, Z', and $U^{(c)}$ are independent, $Z' \stackrel{d}{=} Z'' \stackrel{d}{=} Z$, $U^{(c)\prime} \stackrel{d}{=} U^{(c)}$. It follows that V is a selfdecomposable process.

The proof that (a) implies (c) is as follows. The process V can be extended to a stationary process $\widetilde{V} = \{\widetilde{V}_t : t \in \mathbb{R}\}$ such that $\{\widetilde{V}_t : t \geqslant 0\} \stackrel{d}{=} V$. Define $Y_t = t^{\lambda}\widetilde{V}_{\log t}$, $t > 0$, and $Y_0 = 0$. Then $Y = \{Y_t : t \geqslant 0\}$ is a λ-selfsimilar additive process (see Jeanblanc, Pitman and Yor[23] or Maejima and Sato[25]). If V is a selfdecomposable process, then $\mathcal{L}(Y_t - Y_s)$ is selfdecomposable for $0 \leqslant s < t$ (since we can use $\mathcal{L}(Y_t - Y_s) = \mathcal{L}(t^{\lambda}V_{\log t} - s^{\lambda}V_{\log s})$ for $1 \leqslant s < t$ and $\mathcal{L}(Y_t - Y_s) = \mathcal{L}(c^{-\lambda}(Y_{ct} - Y_{cs}))$ for $0 \leqslant s < t$ and $c > 0$) and thus $\mathcal{L}(Y_t)$ is of class L_1 for $t \geqslant 0$ by Remark 3.14, and it follows that $\mathcal{L}(V_t)$ is of class L_1.

Let us see that (c) implies (b). If $\mathcal{L}(V_t)$ is of class L_1, then $\mathcal{L}(Z_1)$ is selfdecomposable (for a proof see Rocha-Arteaga and Sato[30] Theorem 46) and consequently Z is a selfdecomposable process by the last sentence of Remark 3.14. $\square$

For examples of laws of class L_1, see Akita and Maejima[1].

Proposition 4.3. *Suppose that V is a càdlàg process on $\mathbb{R}^d$. Let*

$$T_t = \int_0^t V_s ds, \qquad t \geqslant 0. \tag{4.3}$$

(i) *If V is infinitely divisible and for each $k \in \mathbb{N}$ there are independent, identically distributed, càdlàg processes $V^{(k,1)}, \dots, V^{(k,k)}$ such that $V \stackrel{d}{=} V^{(k,1)} + \cdots + V^{(k,k)}$, then T is an infinitely divisible process on $\mathbb{R}^d$.*

(ii) *If V is selfdecomposable and for each $c \in (0,1)$ there are independent càdlàg processes V' and $U^{(c)}$ such that $V \stackrel{d}{=} V'$ and $V \stackrel{d}{=} cV' + U^{(c)}$, then T is a selfdecomposable process on $\mathbb{R}^d$.*

Remark 4.4. If V is a stationary OU process, then V satisfies the assumption in (i). If moreover the background driving Lévy process is selfdecomposable, then V satisfies the assumption in (ii). This follows from the proofs of Theorems 4.1 and 4.2. However, we do not know whether $X^{(k,1)}, \dots, X^{(k,k)}$ or $U^{(c)}$ in Theorem 3.6 or 3.7 can always be chosen to be càdlàg when X is càdlàg.

Proof of Proposition 4.3. Since $V_s(\omega)$ is càdlàg, it is measurable in (s,ω) and locally bounded in s for each ω. Thus $\int_0^t V_s(\omega)ds$ exists and belongs to $\mathbb{R}^d$. In general, if V and V' are càdlàg and $V \stackrel{d}{=} V'$, then

$$\left\{\int_0^t V_s ds : t \geqslant 0\right\} \stackrel{d}{=} \left\{\int_0^t V'_s ds : t \geqslant 0\right\}.$$

Now, to see the first assertion, notice that

$$T \stackrel{d}{=} T^{(k,1)} + \cdots + T^{(k,k)}$$

where $T^{(k,l)}_t = \int_0^t V^{(k,l)}_s ds$. Similarly for the second assertion. $\square$

The reason why we have considered integrals of V_s in (4.3) is the following. If Z in (4.1) is a subordinator, then $V = \{V_t : t \geqslant 0\}$ is a nonnegative process. In general, chronometers T of the form (4.3), where $V = \{V_t(\omega) : t \geqslant 0\}$ is a nonnegative stochastic process measurable in (t, ω), are of particular interest in mathematical finance, especially when V is a volatility process. See Geman, Madan and Yor[21], Carr, Geman, Madan and Yor[14], Barndorff-Nielsen and Shephard[6], and references given there. In most cases, V has the interpretation of being the variance process in a stochastic volatility model for the log price of a financial asset, such as a stock or an exchange rate. Some often considered examples are the Heston model and the OU based stochastic volatility models, cf. the above references. Standard examples of stationary OU processes in mathematical finance are the Gamma-OU process and the IG-OU process, in which V_t follows a gamma, respectively an inverse Gaussian distribution.

5. Temporal selfdecomposability of processes

In this section, we introduce a new notion of stochastic processes, which will be called temporal selfdecomposability. Compared with this concept, the selfdecomposability of stochastic processes in Definition 3.1 can be called spatial selfdecomposability, by the property in Theorem 3.7. The class of temporally selfdecomposable processes is larger than the class of Lévy processes. On the other hand, under a slight restriction, temporally selfdecomposable processes are infinitely divisible processes. The notion of additive processes is also between the notion of Lévy processes and that of infinitely divisible processes. But, as we will see, additive processes are not always temporally selfdecomposable, and temporally selfdecomposable processes are not always additive.

Definition 5.1. A stochastic process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is *temporally selfdecomposable* if, for each $c \in (0, 1)$, there exist independent processes $X^{(c)}$ and $U^{(c)}$ on $\mathbb{R}^d$ such that

$$X \stackrel{d}{=} X^{(c)} + U^{(c)} \tag{5.1}$$

and $X^{(c)} = \{X^{(c)}_t : t \geqslant 0\} \stackrel{d}{=} \{X_{ct} : t \geqslant 0\}$.

Theorem 5.2. *A stochastic process $\{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is temporally selfdecomposable if and only if, for any $\tau \in \mathfrak{T}$ and for any $c \in (0, 1)$, there is an $\mathbb{R}^{(\#\tau)d}$-valued random variable $U^{(c,\tau)}$ such that*

$$X_\tau \stackrel{d}{=} X_{c\tau} + U^{(c,\tau)}$$

and $X_{c\tau}$ and $U^{(c,\tau)}$ are independent.

Proof. The "only if" part is trivial. For the proof of the "if" part, mimic the proof of Theorem 3.7. $\square$

Let us show that, in the usual case, temporal selfdecomposability implies infinite divisibility.

Theorem 5.3. *Let $X = \{X_t : t \geqslant 0\}$ be a temporally selfdecomposable process on $\mathbb{R}^d$, stochastically continuous and with $X_0 = 0$ a.s. Then X is infinitely divisible.*

Proof. For any $\tau \in \mathfrak{T}$ and $c \in (0,1)$, we have

$$X_\tau \stackrel{d}{=} X_{c\tau} + U_\tau^{(c)}$$

where $X_{c\tau}$ and $U_\tau^{(c)}$ in the right-hand side are independent. Let

$$f(\tau, z) = E\exp(i\langle z, X_\tau\rangle) \quad \text{and} \quad g(c, \tau, z) = E\exp(i\langle z, U_\tau^{(c)}\rangle)$$

for $z \in \mathbb{R}^{(\#\tau)d}$. It follows that

$$f(\tau, z) = f(c\tau, z) g(c, \tau, z). \tag{5.2}$$

For $a = 0$ we have not defined $a\tau$ (see the last paragraph of Section 2). But, in the following, we understand that $X_{0\tau} = 0$ a. s. in $\mathbb{R}^{(\#\tau)d}$, noting that $X_0 = 0$ a. s.

Step 1. Fix $\tau \in \mathfrak{T}$. We claim that $f(a\tau, z)$ is continuous as a function of $(a, z) \in [0, \infty) \times \mathbb{R}^{(\#\tau)d}$. Let $(a_k, z_k) \in [0, \infty) \times \mathbb{R}^{(\#\tau)d}$ such that $(a_k, z_k) \to (a, z)$ as $k \to \infty$. Let $(a_{k'}, z_{k'})$ be a subsequence of (a_k, z_k). Then

$$\begin{aligned}
|\langle z_{k'}, X_{a_{k'}\tau}\rangle - \langle z, X_{a\tau}\rangle| &\leqslant |\langle z_{k'}, X_{a_{k'}\tau} - X_{a\tau}\rangle| + |\langle z_{k'} - z, X_{a\tau}\rangle| \\
&\leqslant |z_{k'}||X_{a_{k'}\tau} - X_{a\tau}| + |z_{k'} - z||X_{a\tau}| \to 0 \qquad \text{a.s.}
\end{aligned}$$

via a further subsequence $(a_{k''}, z_{k''})$ of $(a_{k'}, z_{k'})$. (Recall that a sequence of random variables W_k converges in probability to a random variable W if and only if any subsequence $W_{k'}$ of W_k contains a further subsequence $W_{k''}$ that converges a.s. to W.) Hence $\langle z_k, X_{a_k\tau}\rangle \to \langle z, X_{a\tau}\rangle$ in probability. It follows that $E\exp(i\xi\langle z_k, X_{a_k\tau}\rangle) \to E\exp(i\xi\langle z, X_{a\tau}\rangle)$ for all $\xi \in \mathbb{R}$. In particular, $f(a_k\tau, z_k) \to f(a\tau, z)$.

Step 2. We claim that $f(a\tau, z) \neq 0$ for any $\tau \in \mathfrak{T}$, $a \in [0, \infty)$, and $z \in \mathbb{R}^{(\#\tau)d}$. Fix $\tau \in \mathfrak{T}$. Suppose that, on the contrary, $f(a_0\tau, z_0) = 0$ for some $a_0 \in [0, \infty)$ and $z_0 \in \mathbb{R}^{(\#\tau)d}$. If $a = 0$ or $z = 0$, then $f(a\tau, z) = 1$. Since $f(a\tau, z)$ is continuous with respect to (a, z) by Step 1, $f(a\tau, z) \neq 0$ in a neighborhood of $(0, 0)$. Hence we can find $(a_0, z_0) \in ([0, \infty) \times \mathbb{R}^{(\#\tau)d}) \setminus \{(0, 0)\}$ such that $f(a_0\tau, z_0) = 0$ and $f(a\tau, z) \neq 0$ for all $(a, z) \in [0, \infty) \times \mathbb{R}^{(\#\tau)d}$ satisfying $a + |z| < a_0 + |z_0|$. We have $a_0 > 0$ and $z_0 \neq 0$. Since

$$0 = f(a_0\tau, z_0) = f(ca_0\tau, z_0) g(c, a_0\tau, z_0) \qquad \text{for } c \in (0, 1)$$

and since $f(ca_0\tau, z_0) \neq 0$, we have $g(c, a_0\tau, z_0) = 0$ for $c \in (0,1)$. Thus, using a general inequality for characteristic functions (Sato (1999) E 6.11), we get

$$1 = \operatorname{Re}(1 - g(c, a_0\tau, z_0)) \leqslant 4 \operatorname{Re}(1 - g(c, a_0\tau, z_0/2)) = 4 \operatorname{Re}\left(1 - \frac{f(a_0\tau, z_0/2)}{f(ca_0\tau, z_0/2)}\right).$$

The last equality is by (5.2) since $f(ca_0\tau, z_0/2) \neq 0$. Letting $c \uparrow 1$, we get a contradiction. This shows that $f(a\tau, z) \neq 0$ for any a and z.

Step 3. Fix $\tau \in \mathfrak{T}$. Let us show that X_τ is infinitely divisible. For each n let $V_{n,j}$, $j = 1, \ldots, n$, be independent random variables such that $V_{n,j} \stackrel{d}{=} U^{(j/(j+1))}_{((j+1)/(n+1))\tau}$. Let $S_n = \sum_{j=1}^n V_{n,j}$. Use (5.2). Then

$$\begin{aligned} E \exp i\langle z, S_n\rangle &= \prod_{j=1}^n g\left(\frac{j}{j+1}, \frac{j+1}{n+1}\tau, z\right) = \prod_{j=1}^n \frac{f(((j+1)/(n+1))\tau, z)}{f((j/(n+1))\tau, z)} \\ &= \frac{f(\tau, z)}{f((1/(n+1))\tau, z)} \to f(\tau, z) \end{aligned}$$

as $n \to \infty$. We claim that $\{V_{n,j} \colon j = 1, \ldots, n\,;\ n = 1, 2, \ldots\}$ is a null array. This will prove the infinite divisibility of X_τ by Khintchine's theorem (Sato[34] Theorem 9.3). Using Step 2 and (5.2), we have

$$\max_{1 \leqslant j \leqslant n} \left| g\left(\frac{j}{j+1}, \frac{j+1}{n+1}\tau, z\right) - 1\right| = \max_{1 \leqslant j \leqslant n} \left| \frac{f(((j+1)/(n+1))\tau, z)}{f((j/(n+1))\tau, z)} - 1\right| \leqslant \frac{A_n}{B_n},$$

where $A_n = \max_{1 \leqslant j \leqslant n} |f(((j+1)/(n+1))\tau, z) - f((j/(n+1))\tau, z)|$ and $B_n = \min_{1 \leqslant j \leqslant n} |f((j/(n+1))\tau, z)|$. For z fixed, B_n is bigger than a positive constant, since $f(a\tau, z)$ is nonzero and continuous in $a \in [0,1]$; A_n tends to 0 as $n \to \infty$, since $f(a\tau, z)$ is uniformly continuous in $a \in [0,1]$. Hence the null array property is shown as in Sato[34] E 12.12. $\square$

Corollary 5.4. *Let X be a process of the type in Theorem 5.3. Then the process $U^{(c)}$ in* (5.1) *is determined uniquely in law for each $c \in (0,1)$. Moreover, $U^{(c)}$ is an infinitely divisible process.*

We call $U^{(c)}$ the c-residual process of X. It should not be confused with $U^{(c)}$ in Theorem 3.7.

Proof of Corollary. Theorem 5.3 shows that the characteristic functions of finite-dimensional marginals of X do not have zero points. Thus the process $U^{(c)}$ in (5.1) is uniquely determined in law by X and c. Let us show that, for each $\tau \in \mathfrak{T}$, $\mathcal{L}(U^{(c)}_\tau)$ is infinitely divisible. As in Sato[34], p. 92, we can choose sequences $\{m_l\}$, $\{n_l\}$ of integers in such a way that $m_l < n_l$, $m_l \to \infty$, and $m_l/n_l \to c$ as $l \to \infty$. Let $V_{n_l,j}$ be as in the proof of Theorem 5.3, and let $W_l = \sum_{j=1}^{m_l} V_{n_l,j}$, $\widetilde{W}_l = \sum_{j=m_l+1}^{n_l} V_{n_l,j}$,

$S_{n_l} = W_l + \widetilde{W}_l$. Then, as before,

$$E\exp i\langle z, W_l\rangle = \frac{f((m_l+1)/(n_l+1))\tau, z)}{f((1/(n_l+1))\tau, z)}.$$

Hence, as $l \to \infty$, $E\exp i\langle z, W_l\rangle \to f(c\tau, z)$ and $E\exp i\langle z, S_{n_l}\rangle \to f(\tau, z)$. It follows that

$$E\exp i\langle z, \widetilde{W}_l\rangle \to \frac{f(\tau, z)}{f(c\tau, z)} = g(c, \tau, z) = E\exp i\langle z, U_\tau^{(c)}\rangle.$$

Since $\{V_{n_l,j}\}$ is a null array, $\widetilde{W}_l$ is a row sum of a null array. Hence $\mathcal{L}(U_\tau^{(c)})$ is infinitely divisible. $\square$

Remark 5.5. Let $X = \{X_t\}$ be a temporally selfdecomposable process on $\mathbb{R}^d$ and V an $\mathbb{R}^d$-valued random variable independent of X. Then, as is easily seen, the process $Y = \{Y_t\}$ defined by $Y_t = V + X_t$ is again temporally selfdecomposable. If the characteristic function of V has a zero point (for example, if V is uniformly distributed on $[0,1]^d$), then Y is not an infinitely divisible process. Thus we cannot dispense with the assumption $X_0 = 0$ a. s. in Theorem 5.3.

Definition 5.6. Let $m = 2, 3, \ldots$. A stochastic process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ is *m-times temporally selfdecomposable* if it is temporally selfdecomposable and, for each $c \in (0,1)$, the c-residual process of X is $(m-1)$-times temporally selfdecomposable, where 1-time temporally selfdecomposable is understood as temporally selfdecomposable. When X is m-times temporally selfdecomposable for all m, we call it *infinitely temporally selfdecomposable.*

We are now going to show that all Lévy processes are infinitely temporally selfdecomposable.

Theorem 5.7. *Let $X = \{X_t \colon t \geqslant 0\}$ be a Lévy process in law on $\mathbb{R}^d$. Then X is temporally selfdecomposable. Furthermore, for each $c \in (0,1)$, the c-residual process $U^{(c)}$ is a Lévy process in law satisfying $\mathcal{L}(U_1^{(c)}) = \mathcal{L}(X_{1-c})$, and thus X is infinitely temporally selfdecomposable.*

Proof. We use Theorem 5.2. Let $\tau = \{t_1, \ldots, t_n\}$ with $0 \leqslant t_1 < t_2 < \cdots < t_n$. Denote $\mu = \mathcal{L}(X_1)$ and $\mu_\tau = \mathcal{L}(X_\tau)$. Then, for $z = (z_j)_{1\leqslant j\leqslant d}$, $z_j \in \mathbb{R}^d$, we have

$$\begin{aligned}
\widehat{\mu}_\tau(z) &= E\exp(i(\langle z_1, X_{t_1}\rangle + \cdots + \langle z_n, X_{t_n}\rangle)) \\
&= E\exp\sum_{j=1}^n i\langle z_j + z_{j+1} + \cdots + z_n, X_{t_j} - X_{t_{j-1}}\rangle \\
&= \widehat{\mu}(z_1 + \cdots + z_n)^{t_1}\widehat{\mu}(z_2 + \cdots + z_n)^{t_2-t_1}\widehat{\mu}(z_3 + \cdots + z_n)^{t_3-t_2}\cdots\widehat{\mu}(z_n)^{t_n-t_{n-1}}.
\end{aligned}$$

where $t_0 = 0$. Thus

$$\widehat{\mu}_\tau(z) = \widehat{\mu}_{c\tau}(z)\widehat{\rho}^{(c,\tau)}(z),$$

where $\rho^{(c,\tau)}(z)$ is given by

$$\widehat{\rho}^{(c,\tau)}(z) = \widehat{\mu}(z_1 + \cdots + z_n)^{(1-c)t_1} \widehat{\mu}(z_2 + \cdots + z_n)^{(1-c)(t_2-t_1)} \cdots \widehat{\mu}(z_n)^{(1-c)(t_n-t_{n-1})}.$$

It follows from this expression that $U^{(c)}$ is a Lévy process in law with $\mathcal{L}(U_1^{(c)}) = \mathcal{L}(X_{1-c})$. The infinite temporal selfdecomposability is obvious, since $U^{(c)}$ is again a Lévy process and we can repeat the argument. $\square$

Remark 5.8. If X is a temporally selfdecomposable process, then for any choice of $0 < s < t$ and $0 < c < 1$, $\mathcal{L}(X_{ct} - X_{cs})$ is a convolution factor of $\mathcal{L}(X_t - X_s)$, since $X_\tau \stackrel{d}{=} X_{c\tau} + U_\tau^{(c)}$ for $\tau = \{s, t\}$ where $X_{c\tau}$ and $U_\tau^{(c)}$ are independent. Using this fact, we can see that an additive process is not always temporally selfdecomposable. Furthermore, a semi-Lévy process defined in Section 2, which is a special case of an additive process, is not always temporally selfdecomposable. Indeed, let X be a semi-Lévy process on $\mathbb{R}$ defined by $X_t = B_{h(t)}$, where $\{B_t\}$ is Brownian motion and $h(t)$ is the continuous function that satisfies $h(0) = 0$ and, for each $n \in \mathbb{Z}_+$, $h'(t) = 1$ for $2n < t < 2n+1$ and $h'(t) = \varepsilon$ for $2n+1 < t < 2n+2$. Assume that $0 < \varepsilon < 1/2$. Then $\mathcal{L}(X_1 - X_{1/2}) = N(0, 1/2)$ is not a convolution factor of $\mathcal{L}(X_2 - X_1) = N(0, \varepsilon)$. Hence X is not temporally selfdecomposable.

Remark 5.9. Let X be an H-selfsimilar process on $\mathbb{R}^d$. Then X is temporally selfdecomposable if and only if X is selfdecomposable. To see this, note that $X_{c\tau} \stackrel{d}{=} c^H X_\tau$ for $\tau \in \mathfrak{T}$ and $c \in (0,1)$. Thus a selfsimilar additive process X is temporally selfdecomposable if and only if $\mathcal{L}(X_1)$ is of class L_1 (see Remark 3.14). Similarly, for fixed $m = 2, 3, ..., \infty$, a selfsimilar additive process X is m-times temporally selfdecomposable if and only if $\mathcal{L}(X_1) \in L_m$. For, by Theorems 3.8 and 3.9, the c-residual process is always selfsimilar additive.

Remark 5.10. A temporally selfdecomposable process is, of course, not necessarily selfdecomposable. In fact, Lévy processes are temporally selfdecomposable as shown in Theorem 5.7 but they are not always selfdecomposable. On the other hand, a self-decomposable process is not necessarily temporally selfdecomposable. The example of Remark 5.8 is such a process.

Remark 5.11. If $X = \{X_t \colon t \geqslant 0\}$ is a temporally selfdecomposable, stationary OU process on $\mathbb{R}^d$, then $X_t = \gamma$ for all t a.s. with some $\gamma \in \mathbb{R}^d$. More generally, let X be a stochastically continuous, temporally selfdecomposable, stationary process on $\mathbb{R}^d$. Then $X_t = X_0$ for all t a.s. Indeed, since $X_t \stackrel{d}{=} X_{ct}$, we get, from (5.1), $E \exp i\langle z, U_t^{(c)}\rangle = 1$ on a neighborhood of $z = 0$. It follows that $U_t^{(c)} = 0$ a.s., and hence $\{X_t\} \stackrel{d}{=} \{X_{ct}\}$. Therefore, for any t_1, t_2, and $\varepsilon > 0$, $P(|X_{t_1} - X_{t_2}| > \varepsilon) = P(|X_{ct_1} - X_{ct_2}| > \varepsilon) \to 0$ as $c \downarrow 0$. This means $X_{t_1} = X_{t_2}$ a.s. If, moreover, X is a stationary OU process, then the equation (4.2) with X in place of V shows that X is independent of itself and hence $X_0 = \text{const}$ a.s.

Theorem 5.12. *Let $V = \{V_t \colon t \geqslant 0\}$ be a process of Ornstein-Uhlenbeck type on $\mathbb{R}^d$ (that is, a solution of* (4.1)*) starting at* 0*. Then V is not temporally selfdecomposable except when $V_t = (1 - e^{-\lambda t})\gamma$ a. s. with some $\gamma \in \mathbb{R}^d$.*

Proof. By (4.2) we have $V_t = \int_0^t e^{-\lambda(t-u)} dZ_{\lambda u}$. Here $\{Z_t\}$ is an arbitrary Lévy process on $\mathbb{R}^d$. Suppose that V is temporally selfdecomposable. Then each component of V is a one-dimensional temporally selfdecomposable process of Ornstein-Uhlenbeck type starting at 0. Hence we may and do assume that $d = 1$. Moreover, we may and do assume that $\lambda = 1$ (consider $V_{t/\lambda}$). Let $\mu = \mathcal{L}(Z_1) = \mu_{(A,\nu,\gamma)}$. What we want to prove is that $A = 0$ and $\nu = 0$. This will show that $Z = t\gamma$ and $V_t = (1 - e^{-t})\gamma$.

The process V is an infinitely divisible process, as the proof of Theorem 4.1 can be modified to this situation. Let $0 < s < t$. Denote the triplet of $\binom{V_s}{V_t}$ by $(\widetilde{A}_{s,t}, \widetilde{\nu}_{s,t}, \widetilde{\gamma}_{s,t})$. Since

$$\begin{pmatrix} V_s \\ V_t \end{pmatrix} = \int_0^t F(u) dZ_u \quad \text{with } F(u) = \begin{pmatrix} 1_{[0,s]}(u) e^{-s+u} \\ e^{-t+u} \end{pmatrix},$$

we have

$$\widetilde{A}_{s,t} = \int_0^t F(u) A F(u)' du, \qquad \widetilde{\nu}_{s,t}(B) = \int_0^t du \int_{\mathbb{R}} 1_B(F(u)x) \nu(dx)$$

for $B \in \mathcal{B}(\mathbb{R}^2)$. Here $F(u)'$ is the transpose of $F(u)$. See Sato (2004). Hence

$$\begin{aligned} \widetilde{A}_{s,t} &= A \int_0^t \begin{pmatrix} 1_{[0,s]}(u) e^{2(-s+u)} & 1_{[0,s]}(u) e^{-s-t+2u} \\ 1_{[0,s]}(u) e^{-s-t+2u} & e^{2(-t+u)} \end{pmatrix} du \\ &= 2^{-1} A \begin{pmatrix} 1 - e^{-2s} & e^{-t}(e^s - e^{-s}) \\ e^{-t}(e^s - e^{-s}) & 1 - e^{-2t} \end{pmatrix}. \end{aligned}$$

Let

$$\rho_{0,s}(C) = \int_0^s du \int_{\mathbb{R}} 1_C(e^u x) \nu(dx), \quad \rho_{s,t}(C) = \int_s^t du \int_{\mathbb{R}} 1_C(e^u x) \nu(dx)$$

for $C \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned} \widetilde{\nu}_{s,t}(B) &= \int_0^s du \int_{\mathbb{R}} 1_B \begin{pmatrix} e^{-s+u} x \\ e^{-t+u} x \end{pmatrix} \nu(dx) + \int_s^t du \int_{\mathbb{R}} 1_B \begin{pmatrix} 0 \\ e^{-t+u} x \end{pmatrix} \nu(dx) \\ &= \int_{\mathbb{R}} 1_B \begin{pmatrix} e^{-s} x \\ e^{-t} x \end{pmatrix} \rho_{0,s}(dx) + \int_{\mathbb{R}} 1_B \begin{pmatrix} 0 \\ e^{-t} x \end{pmatrix} \rho_{s,t}(dx). \end{aligned}$$

Now, for any $c \in (0, 1)$, the c-residual process $U^{(c)}$ is infinitely divisible by virtue of Corollary 5.4. It follows that $\widetilde{A}_{s,t} - \widetilde{A}_{cs,ct}$ is nonnegative-definite and $\widetilde{\nu}_{s,t} - \widetilde{\nu}_{cs,ct} \geqslant 0$. Fix $s > 0$ and choose c such that $2 - e^{-2s} - e^{2cs} < 0$. This is possible because $2 - e^{-2s} - e^{2cs} \to 2(1 - (e^{-2s} + e^{2s})/2) < 0$ as $c \uparrow 1$. Let $t \to \infty$. Then

$$\begin{aligned} &\det(\widetilde{A}_{s,t} - \widetilde{A}_{cs,ct}) \\ &\quad = 4^{-1} A^2 [(e^{-2cs} - e^{-2s})(e^{-2ct} - e^{-2t}) - (e^{-t}(e^s - e^{-s}) - e^{-ct}(e^{cs} - e^{-cs}))^2] \\ &\quad = 4^{-1} A^2 e^{-2ct} (2 - e^{-2s} - e^{2cs} + o(1)), \end{aligned}$$

which is negative for sufficiently large t, unless $A = 0$. Thus A must be zero. As to the Lévy measures, if ν is not identically zero, then the support of $\widetilde{\nu}_{s,t}$ is located on the union of the two straight lines $\left\{\binom{e^{-s}x}{e^{-t}x} : x \in \mathbb{R}\right\}$ and $\left\{\binom{0}{e^{-t}x} : x \in \mathbb{R}\right\}$, while $\widetilde{\nu}_{cs,ct}$ has a positive mass on the straight line $\left\{\binom{e^{-cs}x}{e^{-ct}x} : x \in \mathbb{R}\right\}$ with the origin deleted, which contradicts the fact that $\widetilde{\nu}_{s,t} - \widetilde{\nu}_{cs,ct} \geqslant 0$. Thus ν must be zero. $\square$

We are going to give a class of temporally selfdecomposable processes, which do not have independent increments in general. Recall that $\int_0^\infty f(s)dZ_s$ with Lévy process $\{Z_t\}$ is defined as the limit in probability of $\int_0^t f(s)dZ_s$ as $t \to \infty$ (Sato[35], p. 230). We need a lemma.

Lemma 5.13. *Suppose that $\{Z_t\}$ is a Lévy process on $\mathbb{R}^d$, $\{Z'_t\}$ is an independent copy of $\{Z_t\}$, and that $f(s)$ is a locally bounded measurable function such that $\int_0^\infty f(s)dZ_s$ is definable. Then, for any $c \in (0,\infty)$, $\int_0^\infty f(s)dZ_{cs}$ is definable and, for any $c \in (0,1)$,*

$$\int_0^\infty f(s)dZ_s \stackrel{d}{=} \int_0^\infty f(s)dZ_{cs} + \int_0^\infty f(s)dZ'_{(1-c)s}.$$

Proof. Let $\mu = \mathcal{L}(Z_1)$. Since

$$\int_{t_1}^{t_2} f(s)dZ_{cs} = \int_{ct_1}^{ct_2} f(s/c)dZ_s$$

by Theorem 4.10 of Sato (2004) and for any $z \in \mathbb{R}^d$

$$C_{\int_{t_1}^{t_2} f(s)dZ_{cs}}(z) = \int_{ct_1}^{ct_2} C_\mu(f(s/c)z)ds = c\int_{t_1}^{t_2} C_\mu(f(s)z)ds \to 0$$

as $t_1, t_2 \to \infty$, we see that $\int_0^\infty f(s)dZ_{cs}$ is definable. Let $0 < c < 1$ and let

$$I = \int_0^\infty f(s)dZ_s, \quad I_1 = \int_0^\infty f(s)dZ_{cs}, \quad I_2 = \int_0^\infty f(s)dZ'_{(1-c)s}.$$

Then, for $z \in \mathbb{R}^d$,

$$\begin{aligned}
E\left[e^{i\langle z,I_1+I_2\rangle}\right] &= E\left[e^{i\langle z,I_1\rangle}\right]E\left[e^{i\langle z,I_2\rangle}\right] \\
&= \exp\left\{\int_0^\infty C_\mu(f(s/c)z)ds\right\}\exp\left\{\int_0^\infty C_\mu(f(s/(1-c))z)ds\right\} \\
&= \exp\left\{\int_0^\infty C_\mu(f(s)z)cds + \int_0^\infty C_\mu(f(s)z)(1-c)ds\right\} \\
&= \exp\left\{\int_0^\infty C_\mu(f(s)z)ds\right\} \\
&= E\left[e^{i\langle z,I\rangle}\right].
\end{aligned}$$

This completes the lemma. $\square$

Theorem 5.14. *Suppose that $\{Z_t\}$ is a Lévy process on $\mathbb{R}^d$, $f(s)$ is a locally bounded measurable function on $[0,\infty)$ such that $\int_0^\infty f(s)dZ_s$ is definable. Then, the process $X = \{X_t \colon t \geqslant 0\}$ defined by*

$$X_t = \int_0^\infty f(s)dZ_{ts} \tag{5.3}$$

is infinitely temporally selfdecomposable.

Note that, if $f(s) = 1_{[0,1]}(s)$, then $X_t = Z_t$. Thus the theorem above includes Theorem 5.7.

Proof of theorem. Definability of X_t is given in Lemma 5.13. Further we have

$$X_t = \int_0^\infty f(s/t)dZ_s \qquad \text{a. s.} \tag{5.4}$$

Let $c \in (0,1)$. We claim that

$$\{X_t \colon t \geqslant 0\} \stackrel{d}{=} \left\{\int_0^\infty f(s)dZ_{cts} \colon t \geqslant 0\right\} + \left\{\int_0^\infty f(s)dZ'_{(1-c)ts} \colon t \geqslant 0\right\}, \tag{5.5}$$

where $\{Z'_t\}$ is an independent copy of $\{Z_t\}$. For one-dimensional marginals, this equality in law follows from Lemma 5.13. Let us consider n-dimensional marginals. Let $t_1, t_2, ..., t_n > 0$. For $z_1, z_2, ..., z_n \in \mathbb{R}^d$, we have, using (5.4) and Lemma 5.13,

$$\begin{aligned}
E\exp\left\{i\sum_{j=1}^n \langle z_j, X_{t_j}\rangle\right\} &= E\exp\left\{i\int_0^\infty \sum_{j=1}^n \langle z_j, f(s/t_j)\rangle dZ_s\right\} \\
&= E\exp\left\{i\int_0^\infty \sum_{j=1}^n \langle z_j, f(s/t_j)\rangle dZ_{cs}\right\} E\exp\left\{i\int_0^\infty \sum_{j=1}^n \langle z_j, f(s/t_j)\rangle dZ'_{(1-c)s}\right\} \\
&= E\exp\left\{i\sum_{j=1}^n \langle z_j, \int_0^\infty f(s)dZ_{ct_js}\rangle\right\} E\exp\left\{i\sum_{j=1}^n \langle z_j, \int_0^\infty f(s)dZ'_{(1-c)t_js}\rangle\right\}.
\end{aligned}$$

This shows (5.5). Therefore X is temporally selfdecomposable and the c-residual process is of the same type. Hence X is infinitely temporally selfdecomposable. $\square$

Remark 5.15. A part of Theorem 5.14 can be generalized as follows. Let $Z = \{Z_t\}$ be a temporally selfdecomposable process on $\mathbb{R}^d$, that is, for $c \in (0,1)$, $\{Z_t\} \stackrel{d}{=} \{Z_{ct}\} + \{V_t^{(c)}\}$, where the two processes in the right-hand side are independent. If a function $f(s)$ on $[0,\infty)$ is such that the stochastic integrals $\int_0^\infty f(s)dZ_{ts}$ and $\int_0^\infty f(s/t)dZ_s$ are definable and equal and $\int_0^\infty f(s)dV_{ts}^{(c)}$ and $\int_0^\infty f(s/t)dV_s^{(c)}$ are definable and equal, then the process X defined as in (5.3) is again temporally selfdecomposable.

Remark 5.16. We study more properties of the process $X = \{X_t\} = \{\int_0^\infty f(s)dZ_{ts}\}$ in Theorem 5.14 when Z is a Lévy process on $\mathbb{R}$ with finite second moment. In the following we assume that f is a continuous, decreasing, integrable, nonnegative function on $[0,\infty)$ with $0 < f(0) < \infty$. Then f is also square integrable. A typical example of f in our mind is $f(s) = e^{-s^\alpha}$ with $\alpha > 0$. Let $\mu = \mathcal{L}(Z_1)$ and let m and v be the mean and variance of μ. The cumulant function $C_\mu(z)$ is of class C^2 and we have $m = -iC'_\mu(0)$ and $v = -C''_\mu(0)$.

(i) We have $X_0 = 0$ a. s. by the definition (5.3). For each $t > 0$,

$$C_{X_t}(z) = \int_0^\infty C_\mu(f(s/t)z)ds = t\int_0^\infty C_\mu(f(s)z)ds.$$

Hence there exists a Lévy process $Y = \{Y_t\}$ such that $X_t \stackrel{d}{=} Y_t$ for each $t \geqslant 0$. We have

$$EX_t = EY_t = mt\int_0^\infty f(s)ds, \qquad \mathrm{Var}\,(X_t) = \mathrm{Var}\,(Y_t) = vt\int_0^\infty f(s)^2 ds.$$

(ii) The covariances are as follows:

$$\mathrm{Cov}\,(X_t, X_{t+u}) = vt\int_0^\infty f(s)f(ts/(t+u))ds,$$

$$\mathrm{Cov}\,(Y_t, Y_{t+u}) = vt\int_0^\infty f(s)^2 ds$$

for $t > 0$ and $u \geqslant 0$. Hence the correlation coefficients are as follows:

$$\mathrm{Corr}\,(X_t, X_{t+u}) = \left(\frac{t}{t+u}\right)^{1/2} \frac{\int_0^\infty f(s)f(ts/(t+u))ds}{\int_0^\infty f(s)^2 ds},$$

$$\mathrm{Corr}\,(Y_t, Y_{t+u}) = \left(\frac{t}{t+u}\right)^{1/2}$$

for $t > 0$ and $u \geqslant 0$. We notice that

$$vt\int_0^\infty f(s)^2 ds \leqslant \mathrm{Cov}\,(X_t, X_{t+u}) \uparrow vtf(0)\int_0^\infty f(s)ds,$$

$$1 \leqslant \frac{\mathrm{Corr}\,(X_t, X_{t+u})}{(t/(t+u))^{1/2}} \uparrow \frac{f(0)\int_0^\infty f(s)ds}{\int_0^\infty f(s)^2 ds}$$

as $u \to \infty$ for fixed t and that

$$t^{-1}\mathrm{Cov}\,(X_t, X_{t+u}) \downarrow v\int_0^\infty f(s)^2 ds, \qquad \frac{\mathrm{Corr}\,(X_t, X_{t+u})}{(t/(t+u))^{1/2}} \downarrow 1$$

as $t \to \infty$ for fixed u.

(iii) To examine dependency of increments of X, we investigate increments of the special X with $f(s) = e^{-s}$. Then the definability condition required in Theorem 5.14 is that Z has finite log-moment (see the proof of Theorem 4.1). Our assumption that Z has finite second moment is much stronger than this. We have

$$\mathrm{Cov}\,(X_t, X_{t+u}) = v(t^{-1} + (t+u)^{-1})^{-1}$$

from the expression in (ii). Thus

$$\mathrm{Var}\,(X_{t+1} - X_t) = v2^{-1}(2t+1)^{-1}.$$

Hence X does not have stationary increments and the variance of the 1-increment of X tends to 0. By elementary calculations

$$\mathrm{Cov}\,(X_{t+1}-X_t, X_{t+u+1}-X_{t+u}) = v(2t+1)u^{-2}+o(u^{-2}),$$
$$\mathrm{Corr}(X_{t+1}-X_t, X_{t+u+1}-X_{t+u}) = 2^{3/2}(2t+1)^{3/2}u^{-3/2}+o(u^{-3/2})$$

as $u\to\infty$ for fixed t.

(iv) Assume again that $f(s)=e^{-s}$. Then, the process X has a continuous modification and determines the process Z pathwise. To prove this, we first note that $t^{-1}Z_t\to m$ as $t\to\infty$ a. s. Hence $\int_0^\infty e^{-ts}|Z_s|ds<\infty$ for $t>0$ a. s. Thus, using (5.4) and the integration-by-parts formula in Sato[35], Corollary 4.9, we get

$$t^{-1}X_{t^{-1}} = t^{-1}\int_0^\infty e^{-ts}dZ_s = \int_0^\infty e^{-ts}Z_s ds \qquad \text{a. s.}$$

for each $t>0$. Notice that $\int_0^\infty e^{-ts}Z_s ds$ is continuous in $t>0$ and that $t\int_0^\infty e^{-ts}Z_s ds = \int_0^\infty e^{-s}Z_{s/t}ds$ tends to 0 as $t\to\infty$. It follows that $X=\{X_t\}$ has a continuous modification and, with this modification,

$$t^{-1}X_{t^{-1}} = \int_0^\infty e^{-ts}Z_s ds \quad \text{for all } t>0$$

almost surely. By the uniqueness theorem in Laplace transform theory and by the càdlàg property of Z, we see that the path $t^{-1}X_{t^{-1}}$, $t>0$, determines the path Z_s, $s\geqslant 0$, uniquely.

Remark 5.17. Another type of examples of the infinitely temporally selfdecomposable processes in Theorem 5.14 is provided by $X^1=\{X_t^1\}$ and $X^2=\{X_t^2\}$ given by

$$X_t^1 = \int_0^\infty \log\left|\frac{t-s}{s}\right| dZ_s^1, \qquad X_t^2 = \int_0^\infty \log\left|\frac{t+s}{s}\right| dZ_s^2,$$

where Z^1 and Z^2 are independent, identically distributed, symmetric α-stable Lévy processes on $\mathbb{R}$ with $1<\alpha\leqslant 2$. For $t>0$, $\log|(t\mp s)/s|$ has asymptotics $\mp t/s$ as $s\to\infty$ and $\log(1/s)$ as $s\downarrow 0$. Hence X_t^1 and X_t^2 are definable. Notice that both X^1 and X^2 are $1/\alpha$-selfsimilar. Furthermore, the process $X=X^1+X^2$ is represented as

$$X_t = \int_{-\infty}^\infty \log\left|\frac{t-s}{s}\right| dZ_s,$$

where dZ_s is defined from $Z_s=Z_s^1$ for $s\geqslant 0$ and $Z_s=Z_{-s}^2$ for $s<0$. This X is a $1/\alpha$-selfsimilar symmetric α-stable process with stationary increments, a special case of the log-fractional stable processes introduced by Kasahara, Maejima, and Vervaat[24]. See Embrechts and Maejima[19], Example 3.6.5.

6. Chronometers

By a *chronometer* we mean a real-valued stochastic process $T = \{T_t \colon t \geqslant 0\}$ that starts at 0 and is increasing, stochastically continuous, and càdlàg in the sense that $T_t(\omega)$ is càdlàg in t for all ω. It is not assumed to have independent increments. (In this paper we are using the words *increase* and *decrease* in the sense allowing flatness.)

Suppose we are given a chronometer T and a stochastic process $X = \{X_t(\omega) \colon t \geqslant 0\}$ on $\mathbb{R}^d$, which is càdlàg in the sense that there is Ω_1 with $P(\Omega_1) = 1$ such that $X_t(\omega)$ is càdlàg for all ω in Ω_1. We assume that T and X are independent. Define

$$Y = X \circ T$$

by

$$Y_t(\omega) = \begin{cases} X(T_t(\omega), \omega) = X_{T_t(\omega)}(\omega) & \text{for } \omega \in \Omega_1, \\ 0 & \text{for } \omega \notin \Omega_1. \end{cases}$$

Then Y is a stochastic process. In such a setup we shall refer to X as the *base process* and to Y as the *time-changed process*.

Any increasing Lévy process is a chronometer and such chronometers are known as *subordinators*. A chronometer which is an additive process is called an *additive chronometer*.

Proposition 6.1. *Let $T = \{T_t \colon t \geqslant 0\}$ be a real-valued infinitely divisible process. Then the following conditions are equivalent.*

(a) *For any t_1, t_2 with $0 \leqslant t_1 < t_2$, $P(0 \leqslant T_{t_1} \leqslant T_{t_2}) = 1$.*

(b) *For any positive integer n and for any $\tau = \{t_1, ..., t_n\}$ with $0 \leqslant t_1 < \cdots < t_n$, let A_τ and ν_τ be the Gaussian covariance and Lévy measure of $T_\tau = (T_{t_j})_{1 \leqslant j \leqslant n}$. Then $A_\tau = 0$, $\int_{|x| \leqslant 1} |x| \, \nu_\tau(dx) < \infty$, $\mathrm{Supp}(\nu_\tau) \subset K_\tau$, and the drift γ_τ^0 is in K_τ where K_τ is the cone in $\mathbb{R}^n$ defined by*

$$K_\tau = \{(x_j)_{1 \leqslant j \leqslant n} : 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n\}. \tag{6.1}$$

(c) *Condition* (b) *holds for $n = 2$.*

Proof. By a theorem of Skorohod[36] (or Sato[34] E 22.11), Conditions (a) and (c) are equivalent. Condition (a) is equivalent to

(a′) $P(0 \leqslant T_{t_1} \leqslant T_{t_2} \leqslant \cdots \leqslant T_{t_n}) = 1$ if $0 \leqslant t_1 < \cdots < t_n$.

By the same theorem, (a′) and (b) are equivalent. □

Proposition 6.2. *Let $T = \{T_t \colon t \geqslant 0\}$, with $T_0 = 0$ a.s., be a real-valued stochastic process which is stochastically continuous and satisfies Condition* (a) *of Proposition 6.1. Then there is a chronometer $\widetilde{T} = \{\widetilde{T}_t \colon t \geqslant 0\}$ which is a modification of T.*

Proof. Let

$$\Omega_0 = \{\omega\colon T_0 = 0 \text{ and } T_{t_1}(\omega) \leqslant T_{t_2}(\omega) \text{ for all } t_1, t_2 \in \mathbb{Q}_+ \text{ with } t_1 < t_2\}.$$

Then $P(\Omega_0) = 1$. For all $t \in \mathbb{R}_+$ define $\widetilde{T}_t(\omega) = \inf_{\mathbb{Q}\ni s>t} T_s(\omega)$ on Ω_0 and $\widetilde{T}_t(\omega) = 0$ on $\Omega \setminus \Omega_0$. Then $\widetilde{T}$ has all properties desired. □

Theorem 6.3. *If T is a selfdecomposable chronometer, then, for any $c \in (0,1)$, there is an infinitely divisible chronometer $S^{(c)}$ such that*

$$T \stackrel{d}{=} cT' + S^{(c)}, \tag{6.2}$$

where T' and $S^{(c)}$ are independent and T' is a copy of T.

Proof. Applying Theorem 3.7 to T, denote by $S^{(c)}$ the process $U^{(c)}$ in that theorem. Let $0 \leqslant t_1 < t_2$. Denote $\tau = \{t_1, t_2\}$ and $K_\tau = \{(x_1, x_2)\colon 0 \leqslant x_1 \leqslant x_2\}$. Let ν_τ and $\nu_\tau^{(c)}$ be the Lévy measures of T_τ and $S_\tau^{(c)}$. Then (6.2) implies that

$$\nu_\tau(dx) = \nu_\tau(c^{-1}dx) + \nu_\tau^{(c)}(dx).$$

Since, by Proposition 6.1, $\mathrm{Supp}(\nu_\tau) \subset K_\tau$ and $\int_{|x|\leqslant 1} |x|\nu_\tau(dx) < \infty$, it follows that $\mathrm{Supp}(\nu_\tau^{(c)}) \subset K_\tau$ and $\int_{|x|\leqslant 1} |x|\nu_\tau^{(c)}(dx) < \infty$. Hence $S^{(c)}$ has drift $\gamma_\tau^{0(c)}$; (6.2) implies that

$$\gamma_\tau^0 = c\gamma_\tau^0 + \gamma_\tau^{0(c)}.$$

Thus $\gamma_\tau^{0(c)} \in K_\tau$. Now, by Propositions 6.1 and 6.2, it follows that a modification of $S^{(c)}$ is a chronometer. □

Remark 6.4. Let T be a temporally selfdecomposable chronometer. Then the statement similar to Theorem 6.3 is not true. That is, the c-residual process $U^{(c)}$ of T defined right after Corollary 5.4 does not necessarily have a chronometer modification. For example, let $h(t)$ be the function $h(t) = t$ for $0 \leqslant t \leqslant 1$ and $h(t) = 1 + (t-1)/3$ for $t > 1$, and let $T_t = h(t)$. Then T is trivially a temporally selfdecomposable chronometer and $U_t^{(c)} = h(t) - h(ct)$. Thus $U_1^{(1/2)} = 1/2$ and $U_2^{(1/2)} = 1/3$.

We add a fact showing that the c-residual process of a temporally selfdecomposable chronometer still possesses properties akin to chronometers.

Proposition 6.5. *Let T be a temporally selfdecomposable chronometer. Then, for each $c \in (0,1)$, the c-residual process $U^{(c)}$ is an infinitely divisible process such that $U_t^{(c)} \geqslant 0$ a. s. for each t and, for each pair of $t_1 < t_2$, there is a nonrandom real number $a_{t_1,t_2}^{(c)}$ for which $U_{t_2}^{(c)} \geqslant U_{t_1}^{(c)} - a_{t_1,t_2}^{(c)}$ a. s.*

Proof. By Theorem 5.3 and Corollary 5.4, T and $U^{(c)}$ are both infinitely divisible and $U^{(c)}$ is unique in law. Denote the triplets of T_t and $U_t^{(c)}$ by (A_t, ν_t, γ_t) and $(A_t^{(c)}, \nu_t^{(c)}, \gamma_t^{(c)})$. Denote the drift of T_t by γ_t^0. Then $A_t = A_{ct} + A_t^{(c)}$ and $\nu_t = \nu_{ct} + \nu_t^{(c)}$. Since $A_t = 0$, we have $A_t^{(c)} = 0$. We have $\mathrm{Supp}\,(\nu_t^{(c)}) \subset [0,\infty)$ and $\int_{(0,1]} x\nu_t^{(c)}(dx) < \infty$, since ν_t has the same properties. Hence, $U_t^{(c)}$ has drift $\gamma_t^{0(c)}$ and we have $\gamma_t^0 = \gamma_{ct}^0 + \gamma_t^{0(c)}$. Since γ_t^0 and γ_{ct}^0 are the left extremes of the supports of T_t and T_{ct}, respectively, and since $T_t \geqslant T_{ct}$ a. s., we see that $\gamma_t^0 \geqslant \gamma_{ct}^0$. Hence $\gamma_t^{0(c)} \geqslant 0$. It follows that $U_t^{(c)} \geqslant 0$ a. s. For $0 < t_1 < t_2$, $\mathcal{L}(U_{t_2}^{(c)} - U_{t_1}^{(c)})$ is a convolution factor of $\mathcal{L}(T_{t_2} - T_{t_1})$ as in Remark 5.8. Since $T_{t_2} - T_{t_1} \geqslant 0$ a. s., $U_{t_2}^{(c)} - U_{t_1}^{(c)}$ has triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ satisfying $\widetilde{A} = 0$, $\operatorname{Supp}\widetilde{\nu} \subset [0, \infty)$, and $\int_{(0,1]} x\widetilde{\nu}(dx) < \infty$. It follows that $U_{t_2}^{(c)} - U_{t_1}^{(c)} \geqslant \widetilde{\gamma}$ a. s. $\square$

Remark 6.6. If T is a chronometer of the integral form in the right-hand side of (5.3) with Z being a subordinator and f being nonnegative and decreasing, then, for each $c \in (0, 1)$, the c-residual process $U^{(c)}$ of T has an increasing process modification. This is because the drift of $U_{t_2}^{(c)} - U_{t_1}^{(c)}$ equals $\int_0^\infty (f(s/t_2) - f(s/t_1))(1-c)ds\, \gamma_1^0$, which is nonnegative.

Theorem 6.7. *Let $m \in \mathbb{N} \cup \{\infty\}$. If T is a chronometer of class L_m, then, for any $c \in (0, 1)$, there is a chronometer $S^{(c)}$ of class L_{m-1} such that*

$$T \stackrel{d}{=} cT' + S^{(c)},$$

where T' and $S^{(c)}$ are independent and T' is a copy of T.

Proof. Combine Theorems 3.11 and 6.3 and notice that the law of $S^{(c)}$ is uniquely determined by the law of T and c. $\square$

Example 6.8. Let $\{M(B) \colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ be an $\mathbb{R}^d$-valued homogeneous independently scattered random measure with finite log-moment, where $\mathcal{B}_{\mathbb{R}}^0$ is the class of bounded Borel sets in $\mathbb{R}$. Then for any $H > 0$ we can define an H-selfsimilar additive process T on $\mathbb{R}^d$ by

$$T_t = \begin{cases} \int_{-\infty}^{\log t} e^{Hu} M(du), & t > 0 \\ 0, & t = 0. \end{cases}$$

It is known that $\mu = \mathcal{L}(T_1)$ is selfdecomposable and that any selfdecomposable distribution μ on $\mathbb{R}^d$ can appear in this way. The process T is selfdecomposable if and only if μ is of class L_1, as in Remark 3.14. See Sato[(32)], Maejima and Sato[(25)], and Sato[(35)] for definitions and proofs. If $d = 1$ and $M(B)$ is nonnegative a.s. for every B, then T is an H-selfsimilar additive chronometer, and if moreover $\mu = \mathcal{L}(T_1)$ is of class L_1, then T is selfdecomposable.

7. Inheritance properties under time change

In this section, let $T = \{T_t \colon t \geqslant 0\}$ be a chronometer and $X = \{X_t \colon t \geqslant 0\}$ a base process on $\mathbb{R}^d$ and suppose that they are independent. Define $Y = X \circ T$ as in the second paragraph of Section 6.

Theorem 7.1. *Assume that X is a Lévy process on $\mathbb{R}^d$ and T is infinitely divisible. Then Y is infinitely divisible.*

Proof. Let $C(z) = C_{X_1}(z)$, $z \in \mathbb{R}^d$. For $0 \leqslant t_1 < t_2 < \cdots < t_n$ and $z_1, \dots, z_n \in \mathbb{R}^d$, we have

$$
\begin{aligned}
&E \exp\{i(\langle z_1, Y_{t_1}\rangle + \cdots + \langle z_n, Y_{t_n}\rangle)\}\\
&\quad = E \exp\{i(\langle z_1 + \cdots + z_n, Y_{t_1}\rangle + \langle z_2 + \cdots + z_n, Y_{t_2} - Y_{t_1}\rangle + \cdots + \langle z_n, Y_{t_n} - Y_{t_{n-1}}\rangle)\}\\
&\quad = E\left[\left(E \exp \sum_{j=1}^{n} i\langle z_j + z_{j+1} + \cdots + z_n, X_{s_j} - X_{s_{j-1}}\rangle\right)_{s_j = T_{t_j},\, j=1,\dots,n}\right]\\
&\quad = E\left[\left(\exp \sum_{j=1}^{n} (s_j - s_{j-1}) C(z_j + z_{j+1} + \cdots + z_n)\right)_{s_j = T_{t_j},\, j=1,\dots,n}\right]\\
&\quad = E \exp \sum_{j=1}^{n} a_j (T_{t_j} - T_{t_{j-1}}) = E_1,
\end{aligned}
$$

say, where $a_j = C(z_j + z_{j+1} + \cdots + z_n)$ and $t_0 = 0$. Let $a = (a_j)_{1 \leqslant j \leqslant n}$. Let $\nu_\sharp$ and $\gamma_\sharp^0$ be the Lévy measure and the drift of $(T_{t_j} - T_{t_{j-1}})_{1 \leqslant j \leqslant n}$, respectively, and use (3.3), noting that $\operatorname{Re} a_j \leqslant 0$. Then

$$
E_1 = \exp\left[\int_{\mathbb{R}_+^n} (e^{\langle a, x\rangle} - 1)\nu_\sharp(dx) + \langle \gamma_\sharp^0, a\rangle\right] = E_2,
$$

say. Denote $\tau = \{t_1, \dots, t_n\}$ and define K_τ by (6.1). Let h be the mapping from K_τ onto $\mathbb{R}_+^\tau$ defined by

$$
K_\tau \ni x = (x_j)_{1 \leqslant j \leqslant n} \mapsto h(x) = (x_j - x_{j-1})_{1 \leqslant j \leqslant n} \in \mathbb{R}_+^\tau,
$$

where $x_0 = 0$. Use the Lévy measure ν_τ and the drift γ_τ^0 of $(T_{t_j})_{1 \leqslant j \leqslant n}$ as in Theorem 3.5. Since $\operatorname{Supp}(\nu_\tau) \subset K_\tau$ and $\gamma_\tau^0 \in K_\tau$ by Proposition 6.1, we see that (Sato[34] Proposition 11.10)

$$
\begin{aligned}
E_2 &= \exp\left[\int_{K_\tau} (e^{\langle a, h(x)\rangle} - 1)\nu_\tau(dx) + \langle h(\gamma_\tau^0), a\rangle\right]\\
&= \exp\left[\int_{K_\tau} \left(\exp \sum_{j=1}^{n} a_j (x_j - x_{j-1}) - 1\right)\nu_\tau(dx) + \sum_{j=1}^{n} a_j (\gamma_{t_j}^0 - \gamma_{t_{j-1}}^0)\right]\\
&= E_3,
\end{aligned}
$$

say. For any $k \in \mathbb{N}$, let $\nu_\tau^{(k)} = k^{-1}\nu_\tau$ and $\gamma^{0(k)} = k^{-1}\gamma^0$. Then $\{\nu_\tau^{(k)}\}$ and $\gamma^{0(k)}$ satisfy (a) and (b) of Theorem 3.5 for all $\tau \in \mathfrak{T}$ with K^τ replaced by $\mathbb{R}_+^\tau$ and hence there is an infinitely divisible process $T^{(k)}$ such that, for any $\tau \in \mathfrak{T}$, $T_\tau^{(k)} = f_{\tau\infty} T^{(k)}$ satisfies

$P(T_\tau^{(k)} \in \mathbb{R}_+^\tau) = 1$ and has triplet $(0, \nu_\tau^{(k)}, \gamma_\tau^{0(k)})_0$, where $\gamma_\tau^{0(k)} = f_{\tau\infty}\gamma^{0(k)}$. Notice that $\mathrm{Supp}(\nu_\tau^{(k)}) \subset K_\tau$ and $\gamma_\tau^{0(k)} \in K_\tau$. Now, by Propositions 6.1 and 6.2, there is a chronometer $\widetilde{T}^{(k)}$ which is a modification of $T^{(k)}$. Choose $\widetilde{T}^{(k)}$ such that X and $\widetilde{T}^{(k)}$ are independent. Let $Y^{(k)} = X \circ \widetilde{T}^{(k)}$. Then

$$
\begin{aligned}
E_3 &= \left(\exp\left[\int_{K_\tau}\left(\exp\sum_{j=1}^n a_j(x_j - x_{j-1}) - 1\right)\nu_\tau^{(k)}(dx) + \sum_{j=1}^n a_j(\gamma_{t_j}^{0(k)} - \gamma_{t_{j-1}}^{0(k)})\right]\right)^k \\
&= \left(E\exp(i(\langle z_1, Y_{t_1}^{(k)}\rangle + \cdots + \langle z_n, Y_{t_n}^{(k)}\rangle))\right)^k.
\end{aligned}
$$

Hence Y_τ is infinitely divisible and Y is an infinitely divisible process. $\square$

Remark 7.2. Theorem 7.1 is not true if X is an additive process. It is not true even if X is a semi-Lévy process. To see this, let $h(t)$ be a nonrandom continuous function with $h(0) = 0$. Then $X_t = h(t)$ can be considered as an additive process. Let $X_t = t^2$ and let T be a Poisson process. Then $Y_t = T_t^2$. For each $t > 0$, $\mathrm{Supp}(T_t^2) = \{0, 1, 4, 9, \ldots\}$. If T_t^2 is infinitely divisible, then its law must be a compound Poisson distribution with Lévy measure concentrated on $\mathbb{N}$ and having a positive mass at 1; but then the support of T_t^2 must equal $\mathbb{Z}_+$. Hence, for each $t > 0$, T_t^2 is not infinitely divisible. For an example of a semi-Lévy process having the same property, let $h(t)$ be t^2 for $0 \leqslant t \leqslant 2$, 4 for $2 \leqslant t \leqslant 4$, and $4n + h(t - 4n)$ for $4n \leqslant t \leqslant 4(n+1)$ for $n \in \mathbb{N}$ and let $X_t = h(t)$.

Theorem 7.3. *Assume that X is a strictly α-stable Lévy process with $0 < \alpha \leqslant 2$ on $\mathbb{R}^d$ and T is selfdecomposable. Then Y is selfdecomposable.*

Proof. Let $c \in (0,1)$. Use T' and $S^{(c)}$ in Theorem 6.3. Let $0 \leqslant t_1 < t_2 < \cdots < t_n$. Repeat the argument at the beginning of the proof of Theorem 7.1. Then, with $C(z) = C_{X_1}(z)$,

$$
\begin{aligned}
&E\exp(i(\langle z_1, Y_{t_1}\rangle + \cdots + \langle z_n, Y_{t_n}\rangle)) \\
&\quad = E\exp\sum_{j=1}^n C(z_j + z_{j+1} + \cdots + z_n)(T_{t_j} - T_{t_{j-1}}) = E_1E_2,
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= E\exp\sum_{j=1}^n C(z_j + z_{j+1} + \cdots + z_n)(cT'_{t_j} - cT'_{t_{j-1}}), \\
E_2 &= E\exp\sum_{j=1}^n C(z_j + z_{j+1} + \cdots + z_n)(S_{t_j}^{(c)} - S_{t_{j-1}}^{(c)}).
\end{aligned}
$$

Now use the strict α-stability of X. Then

$$\begin{aligned} E_1 &= E \exp \sum_{j=1}^{n} C(c^{1/\alpha}(z_j + z_{j+1} + \cdots + z_n))(T'_{t_j} - T'_{t_{j-1}}) \\ &= E \exp(ic^{1/\alpha}(\langle z_1, Y_{t_1}\rangle + \cdots + \langle z_n, Y_{t_n}\rangle)). \end{aligned}$$

On the other hand, constructing a copy X' of X independent of X and $S^{(c)}$ and letting $Y^{(c)} = X' \circ S^{(c)}$, we have

$$E_2 = E \exp(i(\langle z_1, Y^{(c)}_{t_1}\rangle + \cdots + \langle z_n, Y^{(c)}_{t_n}\rangle)).$$

In conclusion,

$$Y \stackrel{d}{=} c^{1/\alpha} Y' + X' \circ S^{(c)},$$

where Y', X', $S^{(c)}$ are independent and $Y' \stackrel{d}{=} Y$, $X' \stackrel{d}{=} X$. Since $c^{1/\alpha}$ can be any number between 0 and 1, this shows that Y is selfdecomposable. □

Remark 7.4. Suppose that T is temporally selfdecomposable. Then Y is not necessarily temporally selfdecomposable, even when X is Brownian motion on $\mathbb{R}$. For example, if $h(t)$ is the function given in Remark 5.8 and if $T_t = h(t)$, then T is temporally selfdecomposable but Y is not. However if, for each $c \in (0,1)$, the c-residual process of T has an increasing process modification as in the case of Remark 6.6, then, for any Lévy process X, the process Y is temporally selfdecomposable. The proof of this fact is similar to that of Theorem 7.1.

Unless X is a strictly stable Lévy process, we cannot get the strong conclusion like Theorem 7.3, but at least the following is true.

Proposition 7.5. *Assume that X is a Lévy process on $\mathbb{R}^d$ and T is selfdecomposable. Then, for every $c \in (0,1)$, there is a process $V^{(c)}$ on $\mathbb{R}^d$ such that*

$$\{Y_t \colon t \geqslant 0\} \stackrel{d}{=} \{X(cT_t) + V^{(c)}_t \colon t \geqslant 0\} \tag{7.1}$$

and X, T, and $V^{(c)}$ are independent. Furthermore, in this case, $V^{(c)}_t$ can be represented as $V^{(c)}_t = X'(S^{(c)}_t)$, where $X' \stackrel{d}{=} X$, $S^{(c)}$ is a chronometer such that X, T, X', and $S^{(c)}$ are independent.

Proof. This fact is shown in the proof of Theorem 7.3. □

If $T_t = t$, the property (7.1) is temporal selfdecomposability. Therefore, if $T_t = t$, the first half of Proposition 7.5 is that of Theorem 5.7.

Theorem 7.6. *Let $m \in \mathbb{N} \cup \{0, \infty\}$. Assume that X is a strictly α-stable Lévy process with $0 < \alpha \leqslant 2$ on $\mathbb{R}^d$ and T is of class L_m. Then Y is of class L_m.*

Proof. This is Theorem 7.3 when $m = 0$. The proof for general m is by induction, combining the proof of Theorem 7.3 with Theorem 6.7. The assertion for $m = \infty$ follows from that for m finite. $\square$

The following result reduces to a known property of subordination when T is a subordinator.

Theorem 7.7. *Let $m \in \mathbb{N} \cup \{0, \infty\}$. Assume that X is a strictly α-stable Lévy process with $0 < \alpha \leqslant 2$ on $\mathbb{R}^d$ and that, for each t, $\mathcal{L}(T_t) \in L_m(\mathbb{R})$ (no assumption on the multivariate marginals of T). Then, for each t, $\mathcal{L}(Y_t) \in L_m(\mathbb{R}^d)$.*

Proof. Let $m = 0$. Let $c \in (0, 1)$. For each t we have the decomposition

$$T_t \stackrel{d}{=} cT'_t + S_t^{(c)},$$

where T'_t and $S_t^{(c)}$ are independent and $S_t^{(c)} \geqslant 0$. We can choose $S_t^{(c)}$ so that X and $S_t^{(c)}$ are independent. We have

$$E \exp(i\langle z, Y_t\rangle) = E \exp(C(z)T_t) = E \exp(cC(z)T_t)E \exp(C(z)S_t^{(c)})$$

and

$$E \exp(cC(z)T_t) = E \exp(C(c^{1/\alpha} z)T_t) = E \exp(i\langle z, c^{1/\alpha} Y_t\rangle),$$
$$E \exp(C(z)S_t^{(c)}) = E \exp(i\langle z, X(S_t^{(c)})\rangle).$$

It follows that $\mathcal{L}(Y_t) \in L_0(\mathbb{R}^d)$. For $1 \leqslant m < \infty$, use induction. The assertion for $m = \infty$ follows from the case $m < \infty$. $\square$

References

[1] Akita, K., and Maejima, M. (2002). On certain self-decomposable self-similar processes with independent increments. *Statist. Probab. Letters* **59**, 53–59.

[2] Barndorff-Nielsen, O.E., Blæsild, P., and Schmiegel, J. (2004). A parsimonious and universal description of turbulent velocity increments. *Eur. Phys. J.* **B 41**, 345-363.

[3] Barndorff-Nielsen, O.E., Nicolato, E., and Shephard, N. (2002). Some recent developments in stochastic volatility modelling. *Quantitative Finance* **2**, 11–23.

[4] Barndorff-Nielsen, O.E., Pedersen, J., and Sato, K. (2001). Multivariate subordination, self-decomposability and stability. *Adv. Appl. Probab.* **33**, 160–187.

[5] Barndorff-Nielsen, O.E., and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with Discussion). *J. R. Statist. Soc.* **B 63**, 167–241.

[6] Barndorff-Nielsen, O.E., and Shephard, N. (2005). Impact of jumps on returns and realised variances: econometric analysis of time-deformed Lévy processes. *J. Econometrics* (to appear.)

[7] Barndorff-Nielsen, O.E., and Shephard, N. (2005). *Continuous Time Approach to Financial Volatility.* Cambridge University Press. (To appear.)

[8] Barndorff-Nielsen, O.E., and Shiryaev, A.N. (2005). *Change of Time and Change of Measure.* (In preparation.)

[9] Bertoin, J. (1996). *Lévy Processes.* Cambridge University Press, Cambridge.

[10] Bertoin, J. (1997). Subordinators: Examples and Applications. In P. Bernard (Ed.): *Lectures on Probability Theory and Statistics.* Ecole d'Eté de Probabilities de Saint-Flour XXVII–1997, 1–91.

[11] Bochner, S. (1949). Diffusion equation and stochastic processes. *P. Nat. Acad. Sci.* **85**, 369–370.

[12] Bochner, S. (1955). *Harmonic Analysis and the Theory of Probability.* University of California Press, Berkeley and Los Angeles.

[13] Carr, P., Geman, H., Madan, D.B., and Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *J. Business* **75**, 305–332.

[14] Carr, P., Geman, H., Madan, D.B., and Yor, M. (2003). Stochastic volatility for Lévy processes. *Math. Finance* **13**, 345–382.

[15] Chung, K.L., and Zambrini, J.-C. (2003). *Introduction to Random Time and Quantum Randomness.* World Scientific Publishing, Singapore.

[16] Cont, R., and Tankov, P. (2003). *Financial Modelling with Jump Processes.* Chapman and Hall/CRC Press, London.

[17] Eberlein, E. (2001). Application of generalized hyperbolic Lévy motions to finance. In O.E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.): *Lévy Processes: Theory and Applications.* Birkhäuser Verlag, Basel, 319–337.

[18] Eberlein, E., and Prause, K. (2002). The generalized hyperbolic model: financial derivatives and risk measures. In H. Geman, D. Madan, S. Pliska, T. Vorst (Eds.): *Mathematical Finance–Bachelier Congress 2000.* Springer Verlag, Berlin, 245–267.

[19] Embrechts, P., and Maejima, M. (2002). *Selfsimilar Processes.* Princeton Univ. Press, Princeton.

[20] Geman, H., Madan, D.B., and Yor, M. (2001). Time changes for Lévy processes. *Math. Finance* **11**, 79–96.

[21] Geman, H., Madan, D.B., and Yor, M. (2002). Stochastic volatility, jumps and hidden time changes. *Finance and Stochastics* **6**, 63–90.

[22] Itô, K., and McKean, H.P., Jr. (1965). *Diffusion Processes and Their Sample Paths*, Springer, Berlin.

[23] Jeanblanc, M., Pitman, J., and Yor, M. (2002). Self-similar processes with independent increments associated with Lévy and Bessel processes. *Stoch. Proc. Appl.* **100**, 223–231.

[24] Kasahara, Y., Maejima, M., and Vervaat, W. (1988). Log-fractional stable processes. *Stoch. Proc. Appl.* **30**, 329–339.

[25] Maejima, M., and Sato, K. (2003). Semi-Lévy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein-Uhlenbeck type processes. *J. Math. Kyoto Univ.* **43**, 609–639.

[26] Maejima, M., Sato, K., and Watanabe, T. (2000). Distributions of selfsimilar and semi-selfsimilar processes with independent increments. *Statist. Probab. Letters* **47**, 395–401.

[27] Maejima, M., Suzuki, K., and Tamura, Y. (1999). Some multivariate infinitely divisible distributions and their projections. *Probab. Theory Math. Statist.* **19**, 421–428.

[28] Maruyama, G. (1970). Infinitely divisible processes. *Theory Probab. Appl.* **15**, 1–22.

[29] Pedersen, J., and Sato, K. (2003). Cone-parameter convolution semigroups and their subordination. *Tokyo J. Math.* **26**, 503–525

[30] Rocha-Arteaga, A., and Sato, K. (2003) *Topics in Infinitely Divisible Distributions and Lévy Processes.* Aportaciones Matemáticas, Investigación 17, Sociedad Matemática Mexicana.

[31] Samorodnitsky, G., and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes.* Chapman & Hall, New York.

[32] Sato, K. (1991). Self-similar processes with independent increments. *Probab. Th. Rel. Fields* **89**, 285–300.

[33] Sato, K. (1998). Multivariate distributions with selfdecomposable projections. *J. Korean Math. Soc.* **35**, 783–791.

[34] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Univ. Press, Cambridge.

[35] Sato, K. (2004). Stochastic integrals in additive processes and application to semi-Lévy processes. *Osaka J. Math.* **41**, 211–236.
[36] Skorohod, A.V. (1991). *Random Processes with Independent Increments*. Kluwer Academic Pub., Dordrecht, Netherlands.
[37] Winkel, M. (2001). The recovery problem for time-changed Lévy processes. MaPhySto Research Report 2001-37.