Comments on the book "Lévy Processes and Infinitely Divisible Distribution", I

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M. Maejima asked me how to justify the argument from line -3 of page 95 to line 2 of page 96. It is enough to prove the following fact.

FACT. *Let $h(s)$ be an increasing function on $\mathbb{R}$ such that*

$$h(s+u)-h(s) \geq h(s+u-c)-h(s-c) \tag{1}$$

for all $s \in \mathbb{R}$, $u>0$, and $c>0$. Then $h(s)$ is convex.

PROOF. First we claim that $h(s)$ is continuous. The inequality (1) can be written as follows.

$$h(s+c+u)-h(s+c) \geq h(s+u)-h(s). \tag{2}$$

Also

$$h(s+c)-h(s-u+c) \geq h(s)-h(s-u). \tag{3}$$

Here $s \in \mathbb{R}$, $u>0$, and $c>0$. It follows from (2) that

$$h((s+c)+)-h(s+c) \geq h(s+)-h(s),$$

and hence $h(s+)-h(s)=0$ (otherwise $h(t)$ would be ∞ for $t>s$). It follows from (3) that

$$h(s+c)-h((s+c)-) \geq h(s)-h(s-),$$

and hence $h(s)-h(s-)=0$. Therefore $h(s)$ is continuous. Letting $u=c$ in (1), we obtain

$$h(s+u)-h(s) \geq h(s)-h(s-u), \tag{4}$$

that is,

$$\frac{h(s+u)+h(s-u)}{2} \geq h(s). \tag{5}$$

This and continuity imply convexity, as pp. 71–72 of Hardy, Littlewood, and Pólya "Inequalities" says. But this is seen as follows, if we consider the graph of $h(s)$. The property (4) implies that

$$\frac{h(s+u)-h(s)}{u} \geq \frac{h(s+u)-h(s-u)}{2u} \geq \frac{h(s)-h(s-u)}{u}.$$

Hence, for any $s \in \mathbb{R}$, $u > 0$, and $\lambda = k/2^n$ with $k < 2^n$ (n and k are positive integers), we obtain

$$\frac{h(s+\lambda u) - h(s)}{\lambda u} \le \frac{h(s+u) - h(s+\lambda u)}{(1-\lambda)u}. \tag{6}$$

By continuity this is extended to all $\lambda \in (0,1)$, which is convexity. □

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