This is continuation of the three memos \([m1, m2, m3]\) on the limits of some nested classes induced by iteration of stochastic integral operators.

**Correction.** In Definition of \(L^{\alpha}_\infty(R^d)\) in page 1 of \([m2]\), \(\int_{R^d} x^\mu(dx) < \infty\) should be replaced by \(\int_{R^d} x^\mu(dx) = 0\).

As in \([m3]\) let \(p(u) = u^2 e^{-u}\), and \(g(t) = \int_t^\infty p(u)du\) for \(0 < t \leq \infty\). Let \(t = f(s), 0 \leq s < \infty\), be defined by \(s = g(t), 0 < t \leq \infty\). Define

\[
\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(s)dX_s^{(\mu)}\right).
\]

Let

\[
\mathcal{R}_f^m = \mathcal{R}_f^m(R^d) = \Phi_f^m(\mathcal{D}(\Phi_f^m)), \quad m = 1, 2, \ldots
\]

Then (see p. 2 of \([m1]\))

\[
I(R^d) \supset \mathcal{R}_f^1 \supset \mathcal{R}_f^2 \supset \cdots
\]

We will give characterizations of \(\bigcap_{m=1}^\infty \mathcal{R}_f^m\). We will use Theorem A in \([m1]\) and Theorem B'' in \([m3]\), but we will not use Theorem C'' in \([m3]\).

We write the triplet of \(\mathcal{L}^{\alpha}_\infty(R^d)\) as \((\mathcal{A}_\mathcal{L}^{\alpha}, \nu^{\alpha}, \gamma^{\alpha})\) and the decomposition of the Lévy measure of \(\mu \in L_\infty(R^d)\) in Theorem A in \([m1]\) as \(\Gamma^\alpha(d\beta)\) and \(\lambda_\beta^\mu(d\xi)\).

**Lemma 1.** Let \(\mu \in L_\infty(R^d)\) with \(\Gamma^\mu((0, 1]) = 0\). Let

\[
\Gamma^0(d\beta) = (\Gamma(\beta - 1))^{-1}\Gamma^\mu(d\beta),
\]

\[
\nu^0(B) = \int_{(1,2)} \Gamma^0(d\beta) \int S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr, \quad B \in \mathcal{B}(R^d).
\]

Then, \(\int_a^1 s^{-1}ds \int_{|x|>a} x^\nu^0(dx)\) is convergent in \(R^d\) as \(a \uparrow \infty\) if and only if \(\int_{1<|x|<a} x^\nu^\mu(dx)\) is convergent in \(R^d\) as \(a \uparrow \infty\).

**Proof.** Notice that \(\int_{|x|>1} |x|\nu^0(dx) < \infty\) by Lemma in \([m2]\). We have

\[
\int_1^a s^{-1}ds \int_{|x|>s} x^\nu^0(dx) = \int_1^a s^{-1}ds \int_{(1,2)} \Gamma^0(d\beta) \int S \xi \lambda_\beta^\mu(d\xi) \int_s^\infty r^{-\beta-1} dr
\]

\[
= \int_{(1,2)} (\Gamma(\beta - 1))^{-1}(\beta - 1)^{-1}\Gamma^\mu(d\beta) \int S \xi \lambda_\beta^\mu(d\xi) \int_1^a s^{-\beta} ds
\]

\[
= \int_{(1,2)} (\Gamma(\beta))^{-1}\Gamma^\mu(d\beta) \int S \xi \lambda_\beta^\mu(d\xi) \int_1^a s^{-\beta} ds = I_1\ (say)
\]
and
\[ \int_{1<|x|\leq a} xv^\mu(dx) = \int_{(1,2)} \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi) \int_1^a r^{-\beta} dr = I_2 \] (say).

Hence
\[ I_1 - I_2 = \int_{(1,2)} ((\Gamma(\beta))^{-1} - 1) \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi) \int_1^a r^{-\beta} dr. \]

Since
\[ 0 \leq ((\Gamma(\beta))^{-1} - 1) \int_1^a r^{-\beta} dr \leq \frac{1 - \Gamma(\beta)}{\Gamma(\beta)(\beta - 1)} \leq \text{const} \]
for \( 1 < \beta < 2 \), \( I_1 - I_2 \) is convergent in \( \mathbb{R}^d \) as \( a \uparrow \infty \). Hence the lemma follows. \( \square \)

**Lemma 2.** Let \( \mu \in L_\infty(\mathbb{R}^d) \) with \( \Gamma^\mu((0,1]) = 0 \). Let \( v^0 \) be as in Lemma 1. Then
\[ (26) \quad \mu \in \Phi_f(L_\infty(\mathbb{R}^d) \cap \mathcal{D}(\Phi_f)) \]
if and only if
\[ (27) \quad \{ \int_\varepsilon^\infty \int_{\mathbb{R}^d} \frac{x|x|^2v_x^0(dx)}{1 + |tx|^2} \text{ is convergent in } \mathbb{R}^d \text{ as } \varepsilon \downarrow 0 \}
\]
and the limit equals \( -\gamma^\mu \).

**Proof.** Assume (27). Define \( \mu_0^0 \in I(\mathbb{R}^d) \) by the triplet given by
\[ A^\mu_0 = A^\mu, \quad v^{\mu_0} = v^0, \quad \gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2v^{\mu_0}(dx)}{1 + |tx|^2}. \]
Notice that it follows from the definition of \( v^{\mu_0} \) that \( \int_{|x|>1}|x|v^{\mu_0}(dx) < \infty \) and that \( \mu_0 \in L_\infty(\mathbb{R}^d) \). Using Proposition 2.6 (ii) and (iv) of [S06], we see that \( \mu_0 \in \mathcal{D}(\Phi_f) \) and \( \Phi_f(\mu_0) = \mu \), because
\[ \int_0^\infty (f(s))^2 ds = \int_0^\infty \frac{1}{2} t^2 e^{-t} dt = 1, \]
\[ \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x) v^{\mu_0}(dx) = \int_0^\infty t^{-2} e^{-t} dt \int_{\mathbb{R}^d} 1_B(tx) v^{\mu_0}(dx) \]
\[ = \int_0^\infty t^{-2} e^{-t} dt \int_{(1,2)} (\Gamma(\beta - 1))^{-1} \Gamma^\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(tr\xi) r^{-\beta - 1} dr \]
\[ = \int_0^\infty t^{-2} e^{-t} dt \int_{(1,2)} (\Gamma(\beta - 1))^{-1} \Gamma^\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(s\xi) \xi^{-\beta} r^{-\beta - 1} ds \]
\[ = \int_{(1,2)} \Gamma^\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(s\xi) s^{-\beta - 1} ds = \nu^\mu(B), \]
\[ \lim_{t \to \infty} \int_0^t f(s) ds \left( \gamma^{\mu_0} + \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |f(s)x|^2} - \frac{1}{1 + |x|^2} \right) v^{\mu_0}(dx) \right). \]
The “if” part.

Proof. Let $\Gamma$ be as in the proof of Lemma 2. Recall that $\int_{|x|>1} |x| \nu^0(dx) < \infty$. Then

\[
= \lim_{t \to \infty} \int_0^t f(s) ds \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |f(s)x|^2} - 1 \right) \nu^0(dx) \quad (\text{by the def. of } \gamma^0)
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty t^{-1} e^{-t} dt \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |tx|^2} - 1 \right) \nu^0(dx)
\]

\[
= - \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty t e^{-t} dt \int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |tx|^2} \nu^0(dx) = \gamma^\mu.
\]

Hence (26) holds.

Conversely, assume (26). Then there is $\mu_0 \in L_\infty \cap \mathcal{D}(\Phi_f)$ such that $\Phi_f(\mu_0) = \mu$.

By Proposition 4.1 of [S06], the Lévy measure of $\mu_0$ necessarily equals $\nu^0$. Moreover

\[
\gamma^\mu = \lim_{t \to \infty} \int_0^t f(s) ds \left( \gamma^\mu + \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |f(s)x|^2} - \frac{1}{1 + |x|^2} \right) \nu^0(dx) \right)
\]

\[
= - \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty t e^{-t} dt \int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |tx|^2} \nu^0(dx)
\]

by the same calculus as above, since $\gamma^\mu = - \int_{\mathbb{R}^d} \frac{x|x|^2 \nu^0(dx)}{1 + |tx|^2}$ by virtue of Theorem B". This means (27).

\[\square\]

**Theorem D.** Let $\mu \in I(\mathbb{R}^d)$. Then $\mu \in \bigcap_{m=1}^\infty \mathcal{Y}^m_f$ if and only if $\mu \in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1]) = 0$, and

\[
(28) \quad \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) \left( \frac{1}{\Gamma(\beta-1)} \int_\varepsilon^\infty t^{\beta-2} e^{-t} dt \right) \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_{\beta}(d\xi)
\]

is convergent in $\mathbb{R}^d$ as $\varepsilon \downarrow 0$ and the limit equals $-\gamma^\mu$.

**Proof.** The “if” part. Let $I_\varepsilon$ denote (28). For each $\varepsilon > 0$ finiteness of $I_\varepsilon$ is seen from $(\beta - 1) \Gamma(\beta - 1) = \Gamma(\beta)$ and from

\[
\sup_{\beta \in (1,2]} \int_\varepsilon^\infty t^{\beta-2} e^{-t} dt \leq \int_\varepsilon^1 t^{-1} e^{-t} dt + \int_1^\infty e^{-t} dt < \infty.
\]

Define $\mu_0$ as in the proof of Lemma 2. Recall that $\int_{|x|>1} |x| \nu^0(dx) < \infty$. Then

\[
\int_\varepsilon^\infty t e^{-t} dt \int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |tx|^2} \nu^0(dx)
\]

\[
= \int_\varepsilon^\infty t e^{-t} dt \int_{(1,2)} \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_{\beta}(d\xi) \int_0^\infty \frac{r^3}{1 + t^2 r^2} r^{\beta-1} dr
\]

\[
= \int_\varepsilon^\infty t^{\beta-2} e^{-t} dt \int_{(1,2)} \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_{\beta}(d\xi) \int_0^\infty \frac{s^{\beta-2}}{1 + s^2} ds = I_\varepsilon,
\]
Corollary. (i) If \( \mu \in L_\infty(\mathbb{R}^d) \), \( \Gamma^\mu((0,1]) = 0 \), \( \int_{(1,2)} (\beta - 1)^{-1} \Gamma^\mu(d\beta) < \infty \), and \( \int_{\mathbb{R}^d} \mu(dx) = 0 \), then \( \mu \in \bigcap_{m=1}^\infty \mathcal{R}_f^m \).

(ii) If \( \mu \in L_\infty(\mathbb{R}^d) \), \( \Gamma^\mu((0,1]) = 0 \), \( \int_S \xi \lambda_\beta^\mu(d\xi) = 0 \) for \( \Gamma^\mu \)-a.e. \( \beta \in (1,2) \), and \( \gamma^\mu = 0 \), then \( \mu \in \bigcap_{m=1}^\infty \mathcal{R}_f^m \).
(iii) If \( \mu \in L_\infty(\mathbb{R}^d) \), \( \Gamma^\mu((0, 1]) = 0 \),

\[
(29) \quad \gamma^\mu = - \lim_{a \to \infty} \int_{|x| \leq a} \frac{x|x|^2}{1 + |x|^2} \nu^\mu(dx),
\]  

and there is some \( c \in (0, 1] \) such that

\[
(30) \quad \int_S \xi \lambda^\mu_\beta(d\xi) = 0 \quad \text{for } \Gamma^\mu \text{-a.e. } \beta \in (1, 1 + c),
\]

then \( \mu \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \).

**Proof.** (i) Recall that the conditions \( \Gamma^\mu((0, 1]) = 0 \) and \( \int_{(1, 2)}(\beta - 1)^{-1}\Gamma^\mu(d\beta) < \infty \) imply \( \int_{\mathbb{R}^d}|x|\mu(dx) < \infty \) (see Lemma in \([m2]\)). The same conditions imply that \( I_\varepsilon \) is convergent as \( \varepsilon \downarrow 0 \) and that the limit equals

\[
\begin{align*}
\int_{(1, 2)} \frac{B(\frac{3 - \beta}{2}, \frac{\beta + 1}{2})}{\beta - 1} \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_\beta(d\xi) &= \int_S \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_\beta(d\xi) \int_0^\infty \frac{r^3}{1 + r^2} r^{-\beta - 1} \, dr \\
&= \int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |x|^2} \nu^\mu(dx) = -\gamma^\mu \quad \text{(since } \int_{\mathbb{R}^d} x \mu(dx) = 0). \\
\end{align*}
\]

Hence it follows from Theorem D that \( \mu \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \).

(ii) Since the assumption implies \( I_\varepsilon = 0 \), we can use Theorem D.

(iii) Define \( \mu_1 \) and \( \mu_2 \) by the following:

\[
\begin{align*}
A^\mu_1 &= A^\mu, \quad \Gamma^\mu_1(d\beta) = \Gamma^\mu(d\beta \cap (1, 1 + c)), \quad \lambda^\mu_1 = \lambda^\mu_\beta, \quad \gamma^\mu_1 = 0, \\
A^\mu_2 &= 0, \quad \Gamma^\mu_2(d\beta) = \Gamma^\mu(d\beta \cap [1 + c, 2)), \quad \lambda^\mu_2 = \lambda^\mu_\beta, \quad \gamma^\mu_2 = \gamma^\mu.
\end{align*}
\]

Then \( \mu = \mu_1 * \mu_2 \) and, using (ii), we see that \( \mu_1 \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \). Further, \( \int_{|x| > 1} |x|\nu^\mu_2(dx) < \infty \) and

\[
\int_{|x| \leq a} \frac{x|x|^2}{1 + |x|^2} \nu^\mu(dx) = \int_{(1, 2)} \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_\beta(d\xi) \int_0^a \frac{r^3}{1 + r^2} r^{-\beta - 1} \, dr \\
= \int_{(1+c, 2)} \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_\beta(d\xi) \int_0^a \frac{r^3}{1 + r^2} r^{-\beta - 1} \, dr \\
- \int_{(1+c, 2)} \Gamma^\mu(d\beta) \int_S \xi \lambda^\mu_\beta(d\xi) \int_0^\infty \frac{r^3}{1 + r^2} r^{-\beta - 1} \, dr = \int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |x|^2} \nu^\mu_2(dx)
\]

as \( a \to \infty \). Using the assumption, it follows that

\[
-\gamma^\mu_2 = -\gamma^\mu = \int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |x|^2} \nu^\mu_2(dx).
\]

Hence \( \int_{\mathbb{R}^d} x \mu_2(dx) = 0 \). Hence, using (i), \( \mu_2 \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \). For each integer \( m \geq 1 \), \( \mathcal{R}_f^m \) is closed under convolution. Hence \( \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \) is closed under convolution. Hence it contains \( \mu \). \( \square \)
Theorem E. Let \( \mu \in I(\mathbb{R}^d) \).

(i) There exists \( c \in \mathbb{R}^d \) such that \( \mu * \delta_{-c} \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \) if and only if \( \mu \in L_\infty(\mathbb{R}^d) \), \( \Gamma^\mu((0,1]) = 0 \), and \( \int_{1<|x|\leq a} x \nu^\mu(dx) \) is convergent in \( \mathbb{R}^d \) as \( a \to \infty \).

(ii) If there exists \( c \in \mathbb{R}^d \) such that \( \mu * \delta_{-c} \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \), then this \( c \) is unique.

Proof. (i) The “if” part. Choose \( \nu^0 \) as in Lemma 1. Then \( \int_{|x|>1} |x| \nu^0(dx) < \infty \). Lemma 1 tells us that \( \int_1^a s^{-1}ds \int_{|x|>s} x \nu^0(dx) \) is convergent in \( \mathbb{R}^d \) as \( a \to \infty \). Define \( \mu_0 \) by the triplet

\[
A_{\mu_0} = A_\mu, \quad \nu_{\mu_0} = \nu^0, \quad \gamma_{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |x|^2} \nu_{\mu_0}(dx).
\]

It follows from Theorem B' that \( \mu_0 \in \mathcal{D}(\Phi_f) \cap L_\infty \). Let \( \mu' = \Phi_f(\mu_0) \). Then \( A_{\mu'} = A_{\mu} \) and \( \nu_{\mu'} = \nu^\mu \) (see the proof of Lemma 2). Hence \( \mu * \delta_{-c} = \mu' \in \mathcal{R}_f^1 \) for some \( c \in \mathbb{R}^d \). It follows from Corollary (i) that \( \mu_0 \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \). Hence \( \mu' \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \).

The “only if” part. We assume that \( \mu * \delta_{-c} \in \bigcap_{m=1}^{\infty} \mathcal{R}_f^m \) for some \( c \in \mathbb{R}^d \). Let \( \mu' = \mu * \delta_{-c} \). It follows from Theorem D that \( \mu' \in L_\infty \) and \( \Gamma^\mu'((0,1]) = 0 \). Let \( \mu_0 \in \mathcal{D}(\Phi_f) \) be such that \( \Phi_f(\mu_0) = \mu' \). Then \( \nu_{\mu_0} \) equals \( \nu^0 \) of Lemma 1. It follows from Theorem B' that \( \int_1^a s^{-1}ds \int_{|x|>s} x \nu^0(dx) \) is convergent in \( \mathbb{R}^d \) as \( a \to \infty \). Hence, by Lemma 1, \( \int_{1<|x|\leq a} x \nu^0(dx) \) is convergent in \( \mathbb{R}^d \) as \( a \to \infty \).

(ii) Suppose that both \( \mu_1 = \mu * \delta_{-c_1} \) and \( \mu_2 = \mu * \delta_{-c_2} \) belong to \( \mathcal{R}_f^1 \). Let \( \mu_1^0 \) and \( \mu_2^0 \) be such that \( \Phi_f(\mu_1^0) = \mu_1 \) and \( \Phi_f(\mu_2^0) = \mu_2 \). Since \( A_{\mu_1} = A_{\mu_2} \) and \( \nu_{\mu_1} = \nu_{\mu_2} \), we have \( A_{\mu_1^0} = A_{\mu_2^0} \) and \( \nu_{\mu_1^0} = \nu_{\mu_2^0} \) (see Proposition 4.1 of [S06]). Now it follows from Theorem B' that

\[
\gamma_{\mu_1^0} = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |x|^2} \nu_{\mu_1^0}(dx) = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1 + |x|^2} \nu_{\mu_2^0}(dx) = \gamma_{\mu_2^0}.
\]

Therefore \( \mu_1^0 = \mu_2^0 \) and hence \( \mu_1 = \mu_2 \). It follows that \( c_1 = c_2 \).

\[\square\]

References

[m1] Memo November 29, 2007, from KS.
[m2] Memo December 5, 2007, from KS.
[m3] Memo December 15, 2007, from KS.
Correction

Page 2, line 13, and page 3, line 9:
Replace $\gamma^\mu = -\int_{\mathbb{R}^d} \frac{x|x|^2 \mu^\mu(dx)}{1 + |tx|^2}$ by $\gamma^\mu = -\int_{\mathbb{R}^d} \frac{x|x|^2 \mu^\mu(dx)}{1 + |x|^2}$

Page 3, the last line:
Replace $\int_{\varepsilon}^{\infty} t^{\beta - 2} e^{-t} dt \int_{(1,2)} \frac{\Gamma(\mu d\beta)}{\Gamma(\beta - 1)}$ by $\int_{(1,2)} \frac{\Gamma(\mu d\beta)}{\Gamma(\beta - 1)} \int_{\varepsilon}^{\infty} t^{\beta - 2} e^{-t} dt$

Page 4, line 3: Replace $\Phi(\mu_0)$ by $\Phi_f(\mu_0)$