Memo January 28, 2008, from KS

This is continuation of the three memos [m1, m2, m3] on the limits of some nested classes induced by iteration of stochastic integral operators.

Correction. In Definition of $L_\infty^{(\alpha)\sharp}(\mathbb{R}^d)$ in page 1 of [m2], $\int_{\mathbb{R}^d} x\mu(dx) < \infty$ should be replaced by $\int_{\mathbb{R}^d} x\mu(dx) = 0$.

As in [m3] let $p(u) = u^{-2}e^{-u}$, and $g(t) = \int_t^\infty p(u)du$ for $0 < t \leqslant \infty$. Let $t = f(s)$, $0 \leqslant s < \infty$, be defined by $s = g(t)$, $0 < t \leqslant \infty$. Define

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^{\infty-} f(s)dX_s^{(\mu)}\right). \tag{25}$$

Let

$$\mathfrak{R}_f^m = \mathfrak{R}_f^m(\mathbb{R}^d) = \Phi_f^m(\mathfrak{D}(\Phi_f^m)), \qquad m = 1, 2, \dots .$$

Then (see p. 2 of [m1])

$$I(\mathbb{R}^d) \supset \mathfrak{R}_f^1 \supset \mathfrak{R}_f^2 \supset \cdots .$$

We will give characterizations of $\bigcap_{m=1}^\infty \mathfrak{R}_f^m$. We will use Theorem A in [m1] and Theorem B$''$ in [m3], but we will not use Theorem C$''$ in [m3].

We write the triplet of $\mu \in I(\mathbb{R}^d)$ as $(A^\mu, \nu^\mu, \gamma^\mu)$ and the decomposition of the Lévy measure of $\mu \in L_\infty(\mathbb{R}^d)$ in Theorem A in [m1] as $\Gamma^\mu(d\beta)$ and $\lambda_\beta^\mu(d\xi)$.

Lemma 1. *Let $\mu \in L_\infty(\mathbb{R}^d)$ with $\Gamma^\mu((0,1]) = 0$. Let*

$$\Gamma^0(d\beta) = (\Gamma(\beta-1))^{-1}\Gamma^\mu(d\beta),$$
$$\nu^0(B) = \int_{(1,2)} \Gamma^0(d\beta)\int_S \lambda_\beta^\mu(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Then, $\int_1^a s^{-1}ds\int_{|x|>s} x\nu^0(dx)$ is convergent in $\mathbb{R}^d$ as $a \uparrow \infty$ if and only if $\int_{1<|x|\leqslant a} x\nu^\mu(dx)$ is convergent in $\mathbb{R}^d$ as $a \uparrow \infty$.

Proof. Notice that $\int_{|x|>1} |x|\nu^0(dx) < \infty$ by Lemma in [m2]. We have

$$\begin{aligned}\int_1^a s^{-1}ds\int_{|x|>s} x\nu^0(dx) &= \int_1^a s^{-1}ds\int_{(1,2)}\Gamma^0(d\beta)\int_S \xi\lambda_\beta^\mu(d\xi)\int_s^\infty rr^{-\beta-1}dr\\ &= \int_{(1,2)}(\Gamma(\beta-1))^{-1}(\beta-1)^{-1}\Gamma^\mu(d\beta)\int_S \xi\lambda_\beta^\mu(d\xi)\int_1^a s^{-\beta}ds\\ &= \int_{(1,2)}(\Gamma(\beta))^{-1}\Gamma^\mu(d\beta)\int_S \xi\lambda_\beta^\mu(d\xi)\int_1^a s^{-\beta}ds = I_1 \quad \text{(say)}\end{aligned}$$

and

$$\int_{1<|x|\leqslant a} x\nu^\mu(dx) = \int_{(1,2)} \Gamma^\mu(d\beta) \int_S \xi\lambda^\mu_\beta(d\xi) \int_1^a r^{-\beta}dr = I_2 \quad \text{(say)}.$$

Hence

$$I_1 - I_2 = \int_{(1,2)} ((\Gamma(\beta))^{-1} - 1)\Gamma^\mu(d\beta) \int_S \xi\lambda^\mu_\beta(d\xi) \int_1^a r^{-\beta}dr.$$

Since

$$0 \leqslant ((\Gamma(\beta))^{-1} - 1) \int_1^a r^{-\beta}dr \leqslant \frac{1-\Gamma(\beta)}{\Gamma(\beta)(\beta-1)} \leqslant \text{const}$$

for $1 < \beta < 2$, $I_1 - I_2$ is convergent in $\mathbb{R}^d$ as $a \uparrow \infty$. Hence the lemma follows. □

Lemma 2. *Let $\mu \in L_\infty(\mathbb{R}^d)$ with $\Gamma^\mu((0,1]) = 0$. Let ν^0 be as in Lemma 1. Then*

$$\mu \in \Phi_f(L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)) \tag{26}$$

if and only if

$$\begin{cases} \displaystyle\int_\varepsilon^\infty te^{-t}dt \int_{\mathbb{R}^d} \frac{x|x|^2\nu^0(dx)}{1+|tx|^2} \text{ is convergent in } \mathbb{R}^d \text{ as } \varepsilon \downarrow 0 \\ \text{and the limit equals } -\gamma^\mu. \end{cases} \tag{27}$$

Proof. Assume (27). Define $\mu^0 \in I(\mathbb{R}^d)$ by the triplet given by

$$A^{\mu_0} = A^\mu, \quad \nu^{\mu_0} = \nu^0, \quad \gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2\nu^{\mu_0}(dx)}{1+|tx|^2}.$$

Notice that it follows from the definition of ν^{μ_0} that $\int_{|x|>1} |x|\nu^{\mu_0}(dx) < \infty$ and that $\mu_0 \in L_\infty(\mathbb{R}^d)$. Using Proposition 2.6 (ii) and (iv) of [S06], we see that $\mu_0 \in \mathfrak{D}(\Phi_f)$ and $\Phi_f(\mu_0) = \mu$, because

$$\int_0^\infty f(s)^2 ds = \int_0^\infty t^2 t^{-2} e^{-t}dt = 1,$$

$$\begin{aligned}
\int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu^{\mu_0}(dx) &= \int_0^\infty t^{-2}e^{-t}dt \int_{\mathbb{R}^d} 1_B(tx)\nu^{\mu_0}(dx) \\
&= \int_0^\infty t^{-2}e^{-t}dt \int_{(1,2)} (\Gamma(\beta-1))^{-1}\Gamma^\mu(d\beta) \int_S \lambda^\mu_\beta(d\xi) \int_0^\infty 1_B(tr\xi)r^{-\beta-1}dr \\
&= \int_0^\infty t^{-2}e^{-t}dt \int_{(1,2)} (\Gamma(\beta-1))^{-1}\Gamma^\mu(d\beta) \int_S \lambda^\mu_\beta(d\xi) \int_0^\infty 1_B(s\xi)t^\beta s^{-\beta-1}ds \\
&= \int_{(1,2)} \Gamma^\mu(d\beta) \int_S \lambda^\mu_\beta(d\xi) \int_0^\infty 1_B(s\xi)s^{-\beta-1}ds = \nu^\mu(B),
\end{aligned}$$

$$\lim_{t\to\infty} \int_0^t f(s)ds \left(\gamma^{\mu_0} + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu^{\mu_0}(dx)\right)$$

$$= \lim_{t\to\infty} \int_0^t f(s)ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - 1 \right) \nu^{\mu_0}(dx) \quad \text{(by the def. of } \gamma^{\mu_0})$$
$$= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty t^{-1}e^{-t}dt \int_{\mathbb{R}^d} x \left(\frac{1}{1+|tx|^2} - 1 \right) \nu^{\mu_0}(dx)$$
$$= -\lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty te^{-t}dt \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|tx|^2} \nu^{\mu_0}(dx) = \gamma^\mu.$$

Hence (26) holds.

Conversely, assume (26). Then there is $\mu_0 \in L_\infty \cap \mathfrak{D}(\Phi_f)$ such that $\Phi_f(\mu_0) = \mu$. By Proposition 4.1 of [S06], the Lévy measure of μ_0 necessarily equals ν^0. Moreover

$$\gamma^\mu = \lim_{t\to\infty} \int_0^t f(s)ds \left(\gamma^{\mu_0} + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu^{\mu_0}(dx) \right)$$
$$= -\lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty te^{-t}dt \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|tx|^2} \nu^{\mu_0}(dx)$$

by the same calculus as above, since $\gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2 \nu^{\mu_0}(dx)}{1+|tx|^2}$ by virtue of Theorem B″. This means (27). □

Theorem D. *Let* $\mu \in I(\mathbb{R}^d)$. *Then* $\mu \in \bigcap_{m=1}^\infty \mathfrak{R}_f^m$ *if and only if* $\mu \in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1]) = 0$, *and*

$$(28) \qquad \int_{(1,2)} \frac{B(\frac{3-\beta}{2}, \frac{\beta+1}{2})}{\beta-1} \left(\frac{1}{\Gamma(\beta-1)} \int_\varepsilon^\infty t^{\beta-2}e^{-t}dt \right) \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi)$$

is convergent in $\mathbb{R}^d$ *as* $\varepsilon \downarrow 0$ *and the limit equals* $-\gamma^\mu$.

Proof. The "if" part. Let I_ε denote (28). For each $\varepsilon > 0$ finiteness of I_ε is seen from $(\beta-1)\Gamma(\beta-1) = \Gamma(\beta)$ and from

$$\sup_{\beta\in(1,2)} \int_\varepsilon^\infty t^{\beta-2}e^{-t}dt \leqslant \int_\varepsilon^1 t^{-1}e^{-t}dt + \int_1^\infty e^{-t}dt < \infty.$$

Define μ_0 as in the proof of Lemma 2. Recall that $\int_{|x|>1} |x|\nu^{\mu_0}(dx) < \infty$. Then

$$\int_\varepsilon^\infty te^{-t}dt \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|tx|^2} \nu^0(dx)$$
$$= \int_\varepsilon^\infty te^{-t}dt \int_{(1,2)} \Gamma^{\mu_0}(d\beta) \int_S \xi \lambda_\beta^{\mu_0}(d\xi) \int_0^\infty \frac{r^3}{1+t^2r^2} r^{-\beta-1}dr$$
$$= \int_\varepsilon^\infty t^{\beta-2}e^{-t}dt \int_{(1,2)} \frac{\Gamma^\mu(d\beta)}{\Gamma(\beta-1)} \int_S \xi \lambda_\beta^\mu(d\xi) \int_0^\infty \frac{s^{2-\beta}}{1+s^2}ds = I_\varepsilon,$$

using ([GR] p. 292)

$$\int_0^\infty \frac{s^{2-\beta}}{1+s^2}ds = \frac{B(\frac{3-\beta}{2},\frac{\beta-1}{2})}{2} = \frac{\Gamma(\frac{3-\beta}{2})\Gamma(\frac{\beta-1}{2})}{2\Gamma(1)} = \frac{\Gamma(\frac{3-\beta}{2})\Gamma(\frac{\beta+1}{2})}{\beta-1} = \frac{B(\frac{3-\beta}{2},\frac{\beta+1}{2})}{\beta-1}.$$

Therefore, Lemma 2 says that $\mu \in \Phi_f(L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f))$. Moreover, $\Phi(\mu_0)=\mu$ as in the proof of Lemma 2. The distribution μ_0 satisfies the same condition as μ does. Indeed, $\mu_0 \in L_\infty(\mathbb{R}^d)$, $\Gamma^{\mu_0}((0,1])=0$,

$$\int_{(1,2)} \frac{B(\frac{3-\beta}{2},\frac{\beta+1}{2})}{\beta-1}\left(\frac{1}{\Gamma(\beta-1)}\int_\varepsilon^\infty t^{\beta-2}e^{-t}dt\right)\Gamma^{\mu_0}(d\beta)\int_S \xi\lambda_\beta^{\mu_0}(d\xi)$$

is convergent in $\mathbb{R}^d$ as $\varepsilon\downarrow 0$ by the dominated convergence theorem since

$$\int_{(1,2)}\frac{\Gamma^{\mu_0}(d\beta)}{\beta-1} = \int_{(1,2)}\frac{\Gamma^{\mu}(d\beta)}{\Gamma(\beta)} < \infty,$$

and the limit equals

$$\begin{aligned}&\int_{(1,2)} \frac{B(\frac{3-\beta}{2},\frac{\beta+1}{2})}{\beta-1}\Gamma^{\mu_0}(d\beta)\int_S \xi\lambda_\beta^{\mu_0}(d\xi)\\ &= \int_{(1,2)}\Gamma^{\mu_0}(d\beta)\int_S \xi\lambda_\beta^{\mu_0}(d\xi)\int_0^\infty \frac{r^{2-\beta}dr}{1+r^2} = \int_{\mathbb{R}^d}\frac{x|x|^2\nu^{\mu_0}(dx)}{1+|x|^2} = -\gamma^{\mu_0}.\end{aligned}$$

Therefore we can repeat the procedure above to prove $\mu\in\Phi_f^m(L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f^m))$ for all integer $m\geqslant 1$.

The "only if" part. Assume that $\mu\in\bigcap_{m=1}^\infty \mathfrak{R}_f^m$. Then we can prove $\mu\in L_\infty(\mathbb{R}^d)$ and $\Gamma^\mu((0,1])=0$ in the same way as in the cases $0<\alpha<1$ and $1<\alpha<2$ (see the proofs of Theorems C and C′ in [m1, m2]). The distribution $\mu_0\in\mathfrak{D}(\Phi_f)$ satisfying $\Phi_f(\mu_0)=\mu$ is unique (see Proposition 4.1 of [S06]). For the ν^0 defined in Lemma 1, we have

$$\int_0^\infty ds\int_{\mathbb{R}^d} 1_B(f(s)x)\nu^0(dx) = \nu^\mu(B)$$

as in the proof of Lemma 2. Hence $\nu^{\mu_0}=\nu^0$ and $\mu_0\in L_\infty$. It follows that $\mu\in\Phi_f(L_\infty(\mathbb{R}^d)\cap\mathfrak{D}(\Phi_f))$, that is, (26) holds. Since

$$I_\varepsilon = \int_\varepsilon^\infty te^{-t}dt\int_{\mathbb{R}^d}\frac{x|x|^2}{1+|tx|^2}\nu^0(dx)$$

as above, Lemma 2 says that I_ε is convergent on $\mathbb{R}^d$ as $\varepsilon\downarrow 0$ and the limit is $-\gamma^\mu$. □

Corollary. (i) *If $\mu\in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1])=0$, $\int_{(1,2)}(\beta-1)^{-1}\Gamma^\mu(d\beta)<\infty$, and $\int_{\mathbb{R}^d}x\mu(dx)=0$, then $\mu\in\bigcap_{m=1}^\infty\mathfrak{R}_f^m$.*

(ii) *If $\mu\in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1])=0$, $\int_S\xi\lambda_\beta^\mu(d\xi)=0$ for Γ^μ-a.e. $\beta\in(1,2)$, and $\gamma^\mu=0$, then $\mu\in\bigcap_{m=1}^\infty\mathfrak{R}_f^m$.*

(iii) *If* $\mu \in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1]) = 0$,

$$\gamma^\mu = -\lim_{a\to\infty} \int_{|x|\leqslant a} \frac{x|x|^2}{1+|x|^2}\nu^\mu(dx), \tag{29}$$

and there is some $c \in (0,1]$ *such that*

$$\int_S \xi\lambda^\mu_\beta(d\xi) = 0 \quad \text{for } \Gamma^\mu\text{-a.e. } \beta \in (1, 1+c), \tag{30}$$

then $\mu \in \bigcap_{m=1}^\infty \mathfrak{R}_f^m$.

Proof. (i) Recall that the conditions $\Gamma^\mu((0,1]) = 0$ and $\int_{(1,2)}(\beta-1)^{-1}\Gamma^\mu(d\beta) < \infty$ imply $\int_{\mathbb{R}^d}|x|\mu(dx) < \infty$ (see Lemma in [m2]). The same conditions imply that I_ε is convergent as $\varepsilon \downarrow 0$ and that the limit equals

$$\int_{(1,2)} \frac{B(\frac{3-\beta}{2}, \frac{\beta+1}{2})}{\beta-1}\Gamma^\mu(d\beta)\int_S \xi\lambda^\mu_\beta(d\xi) = \int_{(1,2)}\Gamma^\mu(d\beta)\int_S \xi\lambda^\mu_\beta(d\xi)\int_0^\infty \frac{r^3}{1+r^2}r^{-\beta-1}dr$$
$$= \int_{\mathbb{R}^d}\frac{x|x|^2}{1+|x|^2}\nu^\mu(dx) = -\gamma^\mu \qquad (\text{since } \int_{\mathbb{R}^d} x\mu(dx) = 0).$$

Hence it follows from Theorem D that $\mu \in \bigcap_{m=1}^\infty \mathfrak{R}_f^m$.

(ii) Since the assumption implies $I_\varepsilon = 0$, we can use Theorem D.

(iii) Define μ_1 and μ_2 by the following:

$$A^{\mu_1} = A^\mu, \quad \Gamma^{\mu_1}(d\beta) = \Gamma^\mu(d\beta \cap (1,1+c)), \quad \lambda^{\mu_1}_\beta = \lambda^\mu_\beta, \quad \gamma^{\mu_1} = 0,$$
$$A^{\mu_2} = 0, \quad \Gamma^{\mu_2}(d\beta) = \Gamma^\mu(d\beta \cap [1+c,2)), \quad \lambda^{\mu_2}_\beta = \lambda^\mu_\beta, \quad \gamma^{\mu_2} = \gamma^\mu.$$

Then $\mu = \mu_1 * \mu_2$ and, using (ii), we see that $\mu_1 \in \bigcap_{m=1}^\infty \mathfrak{R}_f^m$. Further, $\int_{|x|>1}|x|\nu^{\mu_2}(dx) < \infty$ and

$$\int_{|x|\leqslant a}\frac{x|x|^2}{1+|x|^2}\nu^\mu(dx) = \int_{(1,2)}\Gamma^\mu(d\beta)\int_S \xi\lambda^\mu_\beta(d\xi)\int_0^a \frac{r^3}{1+r^2}r^{-\beta-1}dr$$
$$= \int_{(1+c,2)}\Gamma^\mu(d\beta)\int_S \xi\lambda^\mu_\beta(d\xi)\int_0^a \frac{r^3}{1+r^2}r^{-\beta-1}dr$$
$$\to \int_{(1+c,2)}\Gamma^\mu(d\beta)\int_S \xi\lambda^\mu_\beta(d\xi)\int_0^\infty \frac{r^3}{1+r^2}r^{-\beta-1}dr = \int_{\mathbb{R}^d}\frac{x|x|^2}{1+|x|^2}\nu^{\mu_2}(dx)$$

as $a \to \infty$. Using the assumption, it follows that

$$-\gamma^{\mu_2} = -\gamma^\mu = \int_{\mathbb{R}^d}\frac{x|x|^2}{1+|x|^2}\nu^{\mu_2}(dx).$$

Hence $\int_{\mathbb{R}^d} x\mu_2(dx) = 0$. Hence, using (i), $\mu_2 \in \bigcap_{m=1}^\infty \mathfrak{R}_f^m$. For each integer $m \geqslant 1$, $\mathfrak{R}_f^m$ is closed under convolution. Hence $\bigcap_{m=1}^\infty \mathfrak{R}_f^m$ is closed under convolution. Hence it contains μ. $\square$

Theorem E. *Let* $\mu \in I(\mathbb{R}^d)$.

(i) *There exists* $c \in \mathbb{R}^d$ *such that* $\mu * \delta_{-c} \in \bigcap_{m=1}^{\infty} \mathfrak{R}_f^m$ *if and only if* $\mu \in L_\infty(\mathbb{R}^d)$, $\Gamma^\mu((0,1]) = 0$, *and* $\int_{1<|x|\leqslant a} x\nu^\mu(dx)$ *is convergent in* $\mathbb{R}^d$ *as* $a \to \infty$.

(ii) *If there exists* $c \in \mathbb{R}^d$ *such that* $\mu * \delta_{-c} \in \bigcap_{m=1}^{\infty} \mathfrak{R}_f^m$, *then this* c *is unique.*

Proof. (i) *The "if" part.* Choose ν^0 as in Lemma 1. Then $\int_{|x|>1} |x|\nu^0(dx) < \infty$. Lemma 1 tells us that $\int_1^a s^{-1}ds \int_{|x|>s} x\nu^0(dx)$ is convergent in $\mathbb{R}^d$ as $a \to \infty$. Define μ_0 by the triplet

$$A^{\mu_0} = A^\mu, \quad \nu^{\mu_0} = \nu^0, \quad \gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu^{\mu_0}(dx).$$

It follows from Theorem B$''$ that $\mu^0 \in \mathfrak{D}(\Phi_f) \cap L_\infty$. Let $\mu' = \Phi_f(\mu_0)$. Then $A^{\mu'} = A^\mu$ and $\nu^{\mu'} = \nu^\mu$ (see the proof of Lemma 2). Hence $\mu * \delta_{-c} = \mu' \in \mathfrak{R}_f^1$ for some $c \in \mathbb{R}^d$. It follows from Corollary (i) that $\mu_0 \in \bigcap_{m=1}^{\infty} \mathfrak{R}_f^m$. Hence $\mu' \in \bigcap_{m=1}^{\infty} \mathfrak{R}_f^m$.

The "only if" part. We assume that $\mu * \delta_{-c} \in \bigcap_{m=1}^{\infty} \mathfrak{R}_f^m$ for some $c \in \mathbb{R}^d$. Let $\mu' = \mu * \delta_{-c}$. It follows from Theorem D that $\mu' \in L_\infty$ and $\Gamma^{\mu'}((0,1]) = 0$. Let $\mu_0 \in \mathfrak{D}(\Phi_f)$ be such that $\Phi_f(\mu_0) = \mu'$. Then ν^{μ_0} equals ν^0 of Lemma 1. It follows from Theorem B$''$ that $\int_1^a s^{-1}ds \int_{|x|>s} x\nu^0(dx)$ is convergent in $\mathbb{R}^d$ as $a \to \infty$. Hence, by Lemma 1, $\int_{1<|x|\leqslant a} x\nu^\mu(dx)$ is convergent in $\mathbb{R}^d$ as $a \to \infty$.

(ii) Suppose that both $\mu_1 = \mu * \delta_{-c_1}$ and $\mu_2 = \mu * \delta_{-c_2}$ belong to $\mathfrak{R}_f^1$. Let μ_1^0 and μ_2^0 be such that $\Phi_f(\mu_1^0) = \mu_1$ and $\Phi_f(\mu_2^0) = \mu_2$. Since $A^{\mu_1} = A^{\mu_2}$ and $\nu^{\mu_1} = \nu^{\mu_2}$, we have $A^{\mu_1^0} = A^{\mu_2^0}$ and $\nu^{\mu_1^0} = \nu^{\mu_2^0}$ (see Proposition 4.1 of [S06]). Now it follows from Theorem B$''$ that

$$\gamma^{\mu_1^0} = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu^{\mu_1^0}(dx) = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu^{\mu_2^0}(dx) = \gamma^{\mu_2^0}.$$

Therefore $\mu_1^0 = \mu_2^0$ and hence $\mu_1 = \mu_2$. It follows that $c_1 = c_2$. □

References

[m1] Memo November 29, 2007, from KS.

[m2] Memo December 5, 2007, from KS.

[m3] Memo December 15, 2007, from KS.

[GR] I.S. Gradshteyn and I.M. Ryzhik (1980). *Table of Integrals, Series, and Products*, Fourth edition, Academic Press, San Diego.

[S06] K. Sato (2006). Two families of improper stochastic integrals with respect to Lévy processes, *ALEA Lat. Am. J. Prob. Math. Stat.* **1**, 47–87.

Correction

Page 2, line 13, and page 3, line 9:
Replace $\gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2 \nu^{\mu_0}(dx)}{1+|tx|^2}$ by $\gamma^{\mu_0} = -\int_{\mathbb{R}^d} \frac{x|x|^2 \nu^{\mu_0}(dx)}{1+|x|^2}$
Page 3, the last line:
Replace $\int_{\varepsilon}^{\infty} t^{\beta-2} e^{-t} dt \int_{(1,2)} \frac{\Gamma^{\mu}(d\beta)}{\Gamma(\beta-1)}$ by $\int_{(1,2)} \frac{\Gamma^{\mu}(d\beta)}{\Gamma(\beta-1)} \int_{\varepsilon}^{\infty} t^{\beta-2} e^{-t} dt$
Page 4, line 3: Replace $\Phi(\mu_0)$ by $\Phi_f(\mu_0)$