Memo December 5, 2007, from KS

This is continuation of Memo November 29, 2007. The limit of another nested class of the ranges of stochastic integrals with respect to Lévy processes is considered.

Lemma. *Let* $0 < \alpha < 2$ *and* $\mu \in L_\infty(\mathbb{R}^d)$*. Then* $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty$ *if and only if* $\Gamma^\mu((0,\alpha]) = 0$ *and* $\int_{(\alpha,2)} (\beta-\alpha)^{-1}\Gamma^\mu(d\beta) < \infty$*.*

This is proved in Step 7 of the proof of Theorem C in Memo Nov. 29. It follows that if $1 < \alpha < 2$, $\mu \in L_\infty(\mathbb{R}^d)$, and $\Gamma^\mu((0,\alpha]) = 0$, then $\int_{\mathbb{R}^d} |x|\mu(dx) < \infty$.

Definition. Let $1 < \alpha < 2$. Let $L_\infty^{(\alpha)\sharp}(\mathbb{R}^d)$ denote the class of $\mu \in L_\infty(\mathbb{R}^d)$ satisfying $\Gamma^\mu((0,\alpha]) = 0$ and $\int_{\mathbb{R}^d} x\mu(dx) < \infty$.

This class $L_\infty^{(\alpha)\sharp}(\mathbb{R}^d)$ is closed under convolution, but is not closed under convergence.

Theorem B′. *Let* $1 < \alpha < 2$*,* $p(u) = u^{-\alpha-1}e^{-u}$*, and* $g(t) = \int_t^\infty p(u)du$ *for* $0 < t \leqslant \infty$*. Let* $t = f(s)$*,* $0 \leqslant s < \infty$*, be defined by* $s = g(t)$*,* $0 < t \leqslant \infty$*. Define*

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^{\infty-} f(s)dX_s^{(\mu)}\right). \tag{16}$$

Then the domain and the range of Φ_f *are as follows:*

$$\begin{aligned}
\mathfrak{D}(\Phi_f) &= \{\mu = \mu_{(A,\nu,\gamma)} \colon \int_{|x|>1} |x|^\alpha \nu(dx) < \infty,\ \gamma = -\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2}\} \\
&= \{\mu = \mu_{(A,\nu,\gamma)} \colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty,\ \int_{\mathbb{R}^d} x\mu(dx) = 0\}, \\
\mathfrak{R}(\Phi_f) &= \{\mu = \mu_{(A,\nu,\gamma)} \colon \nu(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha-1}h_\xi(u)du, \\
&\qquad B \in \mathcal{B}(\mathbb{R}^d), \text{ where } \lambda \text{ is a measure on } S \text{ and } h_\xi(u) \text{ is a function} \\
&\qquad \text{measurable in } \xi \text{ and, for } \lambda\text{-a. e. } \xi, \text{ not identically zero,} \\
&\qquad \text{completely monotone in } u \in (0,\infty), \text{ and } \lim_{u\to\infty} h_\xi(u) = 0, \\
&\qquad \int_{\mathbb{R}^d} x\mu(dx) = 0\}
\end{aligned}$$

Moreover,

$$\mathfrak{D}(\Phi_f) = \{\mu \in I(\mathbb{R}^d) \colon \int_0^\infty |C_\mu(f(s)z)|ds < \infty \text{ for } z \in \mathbb{R}^d\}.$$

This result is in Theorems 2.4 and 4.2 of [S06].

Theorem C′. *Let $f(s)$ and Φ_f be as in Theorem B′. Let*

$$\mathfrak{R}_f^m = \mathfrak{R}_f^m(\mathbb{R}^d) = \Phi_f^m(\mathfrak{D}(\Phi_f^m)), \qquad m = 1, 2, \ldots.$$

Then

$$I(\mathbb{R}^d) \supset \mathfrak{R}_f^1 \supset \mathfrak{R}_f^2 \supset \cdots, \tag{17}$$

$$\bigcap_{m=1}^{\infty} \mathfrak{R}_f^m = L_\infty^{(\alpha)\sharp}(\mathbb{R}^d). \tag{18}$$

Proof. Steps 1, 2, 3, and 4 of the proof of Theorem C do not need any change, except to replace "Theorem B " in Step 4 by "Theorem B′ ".

Step 5. Let m be a positive integer. Let $\mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d)$ and $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} = \mathcal{U}^m(\mu)$. Let us show that $\widetilde{\mu} \in \mathfrak{D}(\Phi_f)$ if and only if $\mu \in \mathfrak{D}(\Phi_f)$. The "if" part is already proved in Step 4, but the following proof shows it again.

Assume that $\mu \in \mathfrak{D}(\Phi_f)$. Then Theorem B′ says that $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ and $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$. The discussion in Step 5 of the proof of Theorem C shows that $\int_{|x|>1} |x|^\alpha \widetilde{\nu}(dx) < \infty$. Further,

$$\begin{aligned}
\widetilde{\gamma} &= \int_0^1 u_m(s)ds \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|u_m(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\\
&= \int_0^1 u_m(s)ds \left(-\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2} + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|u_m(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\\
&= \int_0^1 u_m(s)ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|u_m(s)x|^2} - 1\right)\nu(dx)\\
&= -\int_0^1 ds \int_{\mathbb{R}^d} \frac{u_m(s)x|u_m(s)x|^2}{1+|u_m(s)x|^2}\nu(dx)\\
&= -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\widetilde{\nu}(dx),
\end{aligned}$$

where the last equality comes from the formula $\widetilde{\nu}(B) = \int_0^1 ds \int_{\mathbb{R}^d} 1_B(u_m(s)x)\nu(dx)$. Hence $\widetilde{\mu} \in \mathfrak{D}(\Phi_f)$.

Conversely, assume that $\widetilde{\mu} \in \mathfrak{D}(\Phi_f)$. That is, $\int_{|x|>1} |x|^\alpha \widetilde{\nu}(dx) < \infty$ and $\widetilde{\gamma} = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\widetilde{\nu}(dx)$. Then $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ as in Step 5 of the proof of Theorem C, and the equalities above show that

$$\widetilde{\gamma} = \int_0^1 u_m(s)ds \left(-\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2} + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|u_m(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right).$$

On the other hand

$$\widetilde{\gamma} = \int_0^1 u_m(s)ds \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|u_m(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right).$$

Hence

$$0 = \int_0^1 u_m(s)ds \left(\gamma + \int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2}\right).$$

Since $\int_0^1 u_m(s)ds > 0$, we obtain $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$. Hence $\mu \in \mathfrak{D}(\Phi_f)$.

Step 6. The same as Step 6 of the proof of Theorem C.

Step 7. Let $\mu \in L_\infty(\mathbb{R}^d)$. Then $\mu \in \mathfrak{D}(\Phi_f)$ if and only if

$$\Gamma((0,\alpha]) = 0, \quad \int_{(\alpha,2)} (\beta-\alpha)^{-1}\Gamma(d\beta) < \infty, \quad \text{and} \quad \int_{\mathbb{R}^d} x\mu(dx) = 0, \tag{19}$$

where Γ is the Γ-measure of μ. To show this, use Theorem B′ and Lemma at the top of this memo.

Step 8. If $\mu \in L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)$ and $\Gamma(d\beta)$ and $\lambda_\beta(d\xi)$ are those of μ in Theorem A, and if $\widetilde{\mu} = \Phi_f(\mu)$, then $\widetilde{\mu} \in L_\infty(\mathbb{R}^d)$ with Lévy measure $\widetilde{\nu}$ being

$$\widetilde{\nu}(B) = \int_{(\alpha,2)} \Gamma(\beta-\alpha)\Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr, \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{20}$$

$\int_{\mathbb{R}^d} |x|\widetilde{\mu}(dx) < \infty$, and $\int_{\mathbb{R}^d} x\widetilde{\mu}(dx) = 0$. Recall that $\Gamma(\beta-\alpha) \sim (\beta-\alpha)^{-1}$ as $\beta \downarrow \alpha$. Indeed,

$$\begin{aligned}
\widetilde{\nu}(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) \\
&= \int_0^\infty p(u)du \int_{\mathbb{R}^d} 1_B(ux)\nu(dx) \\
&= \int_0^\infty u^{-\alpha-1}e^{-u}du \int_{(\alpha,2)} \Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty 1_B(ur\xi) r^{-\beta-1} dr \\
&= \int_{(\alpha,2)} \Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty u^{-\alpha-1}e^{-u}du \int_0^\infty 1_B(r'\xi) u^\beta (r')^{-\beta-1} dr' \\
&= \int_{(\alpha,2)} \Gamma(\beta-\alpha)\Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty 1_B(r'\xi)(r')^{-\beta-1} dr',
\end{aligned}$$

that is, (20) holds. It follows from (20) combined with Lemma that $\int_{\mathbb{R}^d} |x|\widetilde{\mu}(dx) < \infty$. To see $\int_{\mathbb{R}^d} x\widetilde{\mu}(dx) = 0$, use Theorem 4.2 of [S06].

Step 9. Let us show that

$$\Phi_f(L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)) = L_\infty^{(\alpha)\,\sharp}(\mathbb{R}^d). \tag{21}$$

It follows from Step 8 that the left-hand side is included in the right-hand side. Let $\mu = \mu_{(A,\nu,\gamma)} \in L_\infty^{(\alpha)\,\sharp}(\mathbb{R}^d)$ with ν represented by $\Gamma(d\beta)$ and $\lambda_\beta(d\xi)$. Let $\mu_0 \in I(\mathbb{R}^d)$

with triplet (A_0, ν_0, γ_0) defined by

$$A_0 = (\Gamma(2-\alpha))^{-1} A,$$
$$\nu_0(B) = \int_{(\alpha,2)} (\Gamma(\beta-\alpha))^{-1}\Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr,$$
$$\gamma_0 = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu_0(dx).$$

Then $\mu_0 \in L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)$ by Step 7. The Lévy measure of $\Phi_f(\mu_0)$ equals ν by virtue of Step 8. Using Proposition 2.6 of [S06], we see that $\Phi_f(\mu_0) = \mu$, since

$$\int_0^\infty f(s)^2 A_0 ds = \Gamma(2-\alpha) A_0 = A$$

and since

$$\begin{aligned}\gamma &= -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu(dx) = -\int_0^\infty f(s)ds \int_{\mathbb{R}^d} \frac{x|f(s)x|^2}{1+|f(s)x|^2} \nu_0(dx) \\ &= \int_0^\infty f(s)ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - 1 \right) \nu_0(dx) \\ &= \int_0^\infty f(s)ds \left(\gamma_0 + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu_0(dx) \right).\end{aligned}$$

Here we have used that $\nu(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu_0(dx)$. This shows that $\mu \in \Phi_f(L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f))$.

Step 10. We claim that, for any positive integer m,

$$\Phi_f^m(L_\infty(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f^m)) = L_\infty^{(\alpha)\sharp}(\mathbb{R}^d). \tag{22}$$

The proof is the same as Step 10 of the proof of Theorem C except to replace $L_\infty^{(\alpha)}$ by $L_\infty^{(\alpha)\sharp}$.

Final step. It follows from Step 10 that $\bigcap_{m=1}^\infty \mathfrak{R}_f^m \supset L_\infty^{(\alpha)\sharp}(\mathbb{R}^d)$. Let us show the converse inclusion. It follows from Step 6 that

$$\bigcap_{m=1}^\infty \mathfrak{R}_f^m \subset \bigcap_{m=1}^\infty U_{m-1}(\mathbb{R}^d) = U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d).$$

Here we have used Jurek's result that $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. Next, we claim that if $\mu \in L_\infty(\mathbb{R}^d) \cap \mathfrak{R}_f^1$, then $\Gamma^\mu((0,\alpha]) = 0$ and $\int_{\mathbb{R}^d} x\mu(dx) = 0$. Indeed, if $\mu \in \mathfrak{R}_f^1$, then μ has mean zero as shown in Theorem B′ and the Lévy measure ν^μ has expression using $\lambda(d\xi)$ and $h_\xi(u)$ in Theorem B′. On the other hand, if $\mu \in L_\infty(\mathbb{R}^d)$, then ν^μ has expression using $\Gamma(d\beta) = \Gamma^\mu(d\beta)$ and $\lambda_\beta(d\xi)$ in Theorem A, which is rewritten as

$$\nu^\mu(B) = \int_S \overline{\lambda}(d\xi) \int_{(0,2)} \Gamma_\xi(d\beta) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr$$
$$= \int_S \overline{\lambda}(d\xi) \int_0^\infty 1_B(r\xi) dr \int_{(0,2)} r^{-\beta-1} \Gamma_\xi(d\beta),$$

where $\overline{\lambda}(d\beta)$ is a probability measure on S and $\Gamma_\xi(d\beta)$ is, for each $\xi \in S$, a measure on $(0,2)$ such that $\int_{(0,2)} (\beta^{-1} + (2-\beta)^{-1}) \Gamma_\xi(d\beta) = \text{const}$ and Γ_ξ is measurable in ξ. In fact, $\Gamma(d\beta)\lambda_\beta(d\xi) = \overline{\lambda}(d\xi)\Gamma_\xi(d\beta)$. Now use the uniqueness of the polar decomposition in Lemma 2.1 of [BMS06]. Thus, if $\mu \in L_\infty(\mathbb{R}^d) \cap \mathfrak{R}_f^1$, then there is a positive finite measurable function $c(\xi)$ such that $\lambda(d\xi) = c(\xi)\overline{\lambda}(d\xi)$ and that, for λ-a.e. ξ,

$$h_\xi(r) = c(\xi)^{-1} r^{\alpha+1} \int_{(0,2)} r^{-\beta-1} \Gamma_\xi(d\beta) = c(\xi)^{-1} \int_{(0,2)} r^{\alpha-\beta} \Gamma_\xi(d\beta).$$

Since $h_\xi(r) \to 0$ as $r \to \infty$, we obtain $\Gamma_\xi((0,\alpha]) = 0$, which implies $\Gamma((0,\alpha]) = 0$. This completes the proof that $\bigcap_{m=1}^\infty \mathfrak{R}_f^m = L_\infty^{(\alpha)\,\sharp}(\mathbb{R}^d)$.